Reading
About
Math

Joseph M. Strayhorn, Jr.

Psychological Skills Press
Wexford, Pennsylvania

Published by:
Psychological Skills Press
www.psyskills.com

Author's email: joestrayhorn@gmail.com

ISBN: 978-1-931773-17-1

The cover photo is in the public domain. It is a panoramic view of part of the Orion Nebula, taken via the Hubble Space Telescope. Credit goes to NASA, ESA, and the Hubble Heritage Team.

Table of Contents

Introduction

This book is meant to help students learn two major academic skills at the same time: reading and mathematics. It's meant to be a supplementary math book. This book certainly looks different from most other math books. When you flip through it, you see many words, in a very high ratio to numbers and other symbols. And the questions the students are asked are like reading comprehension questions, not the usual math textbook sort that asks for a numerical answer. Most are "main idea" questions that ask the student to recognize the central point of the section just read.

Like the other books I've written that are in the format of this book, i.e. a brief section followed by an A or B multiple choice question, the quintessential use of the book comes when people use it together – such as a tutor and a student, taking turns reading the sections aloud. The student answers each question, and the tutor either expresses enthusiastic agreement, or gives an explanation for holding a different opinion, in response to each of the student's answers.

My hypothesis is that words are a more powerful tool in mathematical thinking than most people realize. Here's a sentence fairly randomly selected from this book: "When we write any number, literal or otherwise, next to a parenthesis, even with no multiplication sign, we mean for the number to be multiplied by what's in parentheses." Many students never say sentences such as this. And for many of them, at the beginning, sentences like this are hard to comprehend. But my guess is that with more and more experience of reading mathematical language aloud and selecting accurate summaries of such language, the linguistic structures that mathematicians use will become more and more part of the students' cognitive repertoires.

I'm looking for fuller understanding of mathematics than is possible for most students who do math in a fairly typical way. In this typical way, the student skips reading the explanations in the math book, goes straight to the homework problems, gets in mind an algorithm or procedure that yields the correct answer for the problems, and feels finished when all the problems are done. An understanding of why the procedure works may at times be lacking. I would guess that the memory of the procedure itself is less likely to stick without the fuller understanding of why the procedure works.

Will reading lots of paragraphs about mathematics aloud, and answering comprehension questions about these paragraphs, result in better math skills than an equivalent amount of time spent working more problems? Only time, experience, and careful observation will tell!

It’s hard to imagine that reading this book aloud and answering the comprehension questions could fail to give good practice in reading comprehension, provided that the text is neither too hard nor too easy for the student.

As with all the other books I’ve written in this format, it’s necessary to caution the tutor not to think, “The student is getting all the questions right. Therefore this must be too easy. Or the student must already know everything in here.” That’s not how it works. If the student is tuned in and paying attention to what he or she is reading, then a very high fraction of questions should be answered correctly; that’s how programmed instruction works. If the student misses a very high fraction, the book is probably over the student’s ideal level of difficulty.

Of course, this book can be read by someone independently. But the more experience I have with the format of taking turns reading, the student responding to questions, and the tutor responding to the response, the more I find that it’s a pleasant way for people to be with each other and spend time together. Even if the student can master the book alone, making a social activity out of mastering it together can be satisfying for both tutor and student. The social rhythm of working together on it, preferably on a daily schedule, also in my experience makes it much more likely that it will be completed.

Chapter 1: Numbering Systems and Sets

Why numbers are important

1. Why is math so useful? Because we use numbers very often. Almost every day we need to know how many or how much of something we are dealing with.

A doctor sees a patient and does some tests. The test results come back in numbers. The doctor has to figure out what the numbers mean for the patient. Then perhaps the doctor gives the person some medicine. If the person takes too much, the medicine will be harmful or deadly. If the person takes too little, the medicine will not do any good. But just the right amount of medicine will help the person get well. It takes the work of lots of people – those who developed the drug, those who make the pills, the doctor, and often the patient also, among others -- to make sure that just the right amount of medicine is given.

The main point of this section is that

A. Taking too much medicine can be harmful,
or
B. Medicine is one example of a type of work where math is very important?

2. Suppose a builder is making a bridge. If there is not enough steel at the right place, the bridge may fall down. If there is just the right amount, the bridge will be strong. Someone has to use math very carefully to figure out exactly how a bridge should be built, in such a way that it will be safe. The same thing goes for building an airplane, a car, a roller coaster, a house, or almost anything else.

The main point of this paragraph is that

A. Lots of math calculations go into building things,
or
B. When bridges fall down, may people can get hurt.

3. Suppose you have a business. Someone asks you, “How long will it take you to do this job? And how much will it cost?” You will use math to figure out the correct answers to this question. It you tell the person, “It will take three weeks,”

and it really takes three months, the person will likely be mad. If you tell the person, “I will charge you three thousand dollars,” but you haven’t calculated the cost well enough, and the job ends up costing you eight thousand dollars, you have worked hard to lose money! Math can make the difference between your business’s succeeding or failing!

This section gave a reason for a business person to learn math. What was it?

A. So that the business person can figure out how much tax should be paid,
or
B. So that the business person can figure out how long projects will take and how much they will cost.

4. Suppose that there is some chemical that someone is putting onto the ground or into a river. How harmful is this chemical? We use numbers to think about how many people get sick as a result of the chemical. We also use numbers to think about how much of the chemical is released, and to say how much of the chemical is present in a certain amount of water in the river. We use numbers to say how the chemical changes the number of fish and frogs living in the river.

The main point of this section is that

A. Numbers and math help people who are trying to create a safe and nonpolluted environment,
or
B. The same chemicals that can hurt people can also hurt frogs and fish.

5. Suppose that someone has a new way of teaching people to be happy and productive. The person wants to know how helpful the new way of teaching is. Figuring this out takes math, also. The person has to find a way of measuring how much people were helped, and expressing that result in numbers. If a bunch of people get the new teaching method, and a bunch of other people don’t, then the researcher uses math to compare the two groups. In this way we find out whether the things we think are good and useful to do with people really are good and useful.

These are just a few of the thousands of uses of math that we could give.

The section above spoke about how math helps with

A. figuring out which methods of teaching people are most useful,
or
B. figuring out whether one drug is better than another?

We get numbers by counting or measuring

6. Sometimes we find out how many things there are by counting them. How many people are in a family? How many books do I own? How many dollar bills do I have? To answer these questions, we take the things and count them.

But how can you count how much a person weighs, or how tall someone is? How can you count how bright a light is, or how fast a person can run? For questions like this, we get numbers by measuring.

The point of this section is that

A. there are two ways of getting numbers to describe the world: counting, and measuring,
or
B. counting gives an exact number, whereas measuring is never completely exact.

7. We name a certain amount of something a unit, and then we can count how many units it takes to be the same as the thing we are trying to measure. For example, we are trying to measure how tall someone is. We call our unit of measurement the centimeter, and then we can count how many centimeters we have to put together to equal how tall the person is.

Or if we want to measure how long something takes, we call our unit of measure the second, and we count how many seconds something takes to happen.

So measuring means figuring out some sort of unit and then counting how many of that unit fit the thing we are trying to measure.

The point of this section is that

A. measurements all have to do with the basic ideas of length, mass, and time,
or

B. we get numbers from measurement by deciding on a standard unit, and counting how many of that unit are the same as what we're trying to measure.

Sets

8. A set is any group of things, or people, or places, or numbers, or anything else. Suppose three people form a singing group. We say that the set of people in the group has three members. Sometimes we can best tell about a set by listing all its members. We could say that a certain set of people in a singing group is {John, Molly, Sam}.

Here is another set: the set of all people who have brown eyes. I don't know how many members there are in that set, but there must be billions of them. In this example, listing all the members of the set would be close to impossible. But by saying something like, "The set of all X, such that X is a person with brown eyes," we can define who's in our set and who isn't. We are giving a rule rather than giving a list. The rule is that if you have brown eyes, you're in the set, and if you don't, you're not. So there are two ways of defining a set: with a rule, and with a list.

This section made the point that

A. sometimes it's impossible to define a set by listing all its members, but it is possible to give a rule that says who is in and who is out,
or
B. when you give a rule for a set, you have to make it very clear and specific.

9. Here is another set: the set of all people who are still living at age 200. There are no members of that set, because no one lives that long. A set with no members is called "the null set" or "the empty set."

Here is a set of letters: {a, c, d, f}.
Here is a set of numbers: {3, 4, 5}.
Here is a set of people: all people who are over 5 feet tall.
Here is another example of the empty set: all people who are over 15 feet tall.

One point made in the section above is that

A. the empty set has no members,
or
B. the set containing all that are in either of two sets is called the “union” of the two sets.

10. Suppose that we have two sets. One is a list of all the people in my neighborhood who are learning Spanish: {Mary, Peter, Sally, Ralph}. The second is a list of all the people who are learning world history: {Mary, Peter, Jeff, Cindy}. Suppose I want to have a party for anyone in my neighborhood who is studying Spanish OR world history. How many people would I invite? There would be six people: {Mary, Peter, Sally, Ralph, Jeff, Cindy}. The list of people who are in one set OR the other is called the *union* of the two sets. If the first set is A and the second is B, the set of people who are invited to the party are A union B.

The point of this section is that

A. The union of two sets is the list of those who are members of the first OR the second.
or
B. A subset is a set, all of whose members are in another set.

11. Suppose that {Mary, Peter, Sally, Ralph} are learning Spanish, and {Mary, Peter, Jeff, Cindy} are learning world history. Suppose that I want to have a party and invite only the people who are learning BOTH Spanish AND world history. Now whom would I invite? A smaller group – just Mary and Peter. The set of people who are in BOTH one set AND the other is called the *intersection* of the two sets. If the first set is A and the second B, the people who are in both sets are in a set called “A intersect B.”

Which of the following is the intersection of the set of women and the set of doctors?

A. The set of people who are either doctors OR women,
or

B. The set of people who are BOTH doctors AND women, that is, all female doctors.

12. We use these ideas of intersections and unions of sets often when we are searching for articles using a computer. Suppose we are interested in learning about drug treatment of the disease called leprosy. We find out that leprosy is also called Hansen's disease. One of the first things we might do is to lump together all the articles that are about leprosy OR Hansen's disease. We take the union of the two sets.

Then we find a set of articles that are about drug treatment, of any illness at all. There are thousands of them.

Now we tell the computer to find the articles that are about BOTH drug treatment AND about leprosy or Hansen's disease. The computer finds the intersection of the drug treatment set, with the union of the leprosy and Hansen's disease sets! These turn out to be just the articles we want.

The purpose of this section is to

A. help you learn more about the disease called leprosy or Hansen's disease,
or
B. to give an example of how, in searching for articles, we use the ideas of unions and intersections of sets.

Subsets

13. Sometimes we speak of one set as being a *subset* of another. What does this mean? Suppose you start with a group of things. Then you pick out some of those things. For example, you start with the set of all dogs. Then you pick out the set of collies from that set of dogs. We say that the set of collies is a *subset* of the set of dogs. Every collie is also a dog.

Let's think about some other subsets. All girls are people. So the set of girls is a subset of the set of people. All the girls are also people.

Socks are a subset of clothes. Pianos are a subset of musical instruments. Wrenches are a subset of tools. Fords are a subset of cars.

If all the members of the first set are also members of the second, the first is a subset of the second.

This section teaches us that if all computers are machines, then

A. the set of computers is a subset of the set of machines,
or
B. the set of machines is a subset of the set of computers.

14. Here is a set of letters: {A, B, C, D, E, F, G}.

Here is a second set of letters: {A,B}.

Is the second set of letters a subset of the first? Yes, because every member of the second set is also a member of the first set.

Here is a set of numbers: {1, 2, 3, 4, 5}.

Here is a second set of numbers: {1, 8, 9}.

Is the second set of numbers a subset of the first? No, because not every member of the second set is also a member of the first set. Eight and nine are not in the first set of numbers.

This section gives more examples of the fact that

A. sets are useful when solving problems about probability,
or
B. when one set has every member included in a second set, the first is a subset of the second.

15. When one set is a subset of another, their intersection and union come out in a special way each time. Collies are a subset of dogs. What is the union of these two sets? It's those things that are either collies OR dogs. But that's just the set of dogs. What's the intersection of those sets? It's those things that are both collies AND dogs. But that's just the set of collies. So the intersection came out to be the subset, and the union came out to be the larger set.

Let's go through it one more time. Peas are a subset of vegetables. The intersection of peas and vegetables is just the set of peas. The union of peas and vegetables is just the set of vegetables. So the intersection is the subset, and the union is the larger set.

Why do you think the author includes this section?

A. Because it provides practice in thinking about subsets, unions, and intersections, and lets us check our understanding of those ideas.
or
B. Because people very frequently ask each other things like, “What is the intersection of the set of peas with the set of vegetables?”

Matching one-to-one

16. Speaking of food: Suppose we have three buns. We also have three veggie burgers.

Now imagine that we put one burger on each bun. We match up the buns with the burgers so that one burger goes with each bun This is called one-to-one matching. It isn’t one-to-one matching if we pair up two burgers with one bun. It isn’t one-to-one matching if we leave a bun without a burger. You make each member of the set of pieces of buns match up with one member of the set of burgers.

What’s the better summary of the idea of this section?

A. It’s called one-to-one matching when you pair up one of one set, with one of another set.
or
B. When having burgers, each person should get just one burger and just one bun.

17. dog dog dog dog

Suppose someone asked someone to put up one finger for each time the word “dog” is printed above.

Someone could possibly do this even if he didn’t know how to count yet, if he understood one-to-one matching (and if he knew when one word stopped and the next one started). He would start at one end of the list, and go to the other, and put up one finger each time he saw the word. He would be holding 4 fingers up, even if he didn’t know the word for “four.”

Here are some other one-to-one matching tasks. Suppose people are having soup. Someone puts one spoon next to each of a few bowls on a table. If you put two spoons next to a few bowls, and no spoons next to a few bowls, that’s not correct one-to-one matching, is it? You have to put exactly one spoon next to each bowl.

Suppose you then are to put a cup of soup in each bowl. If you leave some bowls empty, and make other bowls overflow onto the table, that's not correct one-to-one matching. The people getting the soup would not be pleased.

All the examples in this section have to do with

A. pairing one of one thing with one of another
or
B. creating subsets.

18. Here's another one-to-one matching task. Someone draws some u's. The student's job is to draw one p to go with each u. If my student puts one p underneath each u, the student has done one-to-one matching correctly

u u u u u
p p p p p

If the student were to put one p beside each u, the student also would have been doing it correctly.

up up up up up

But if you did it as below, that wouldn't be correct!

u u u u u
pp p ppp p

When you do one-to-one matching correctly,

A. there are more of one type of thing than of the other,
or
B. each thing has exactly one partner that's the other type of thing?

Counting is really a one-to-one matching task

19. Suppose that someone asked you to start at the number one and match the counting numbers, in order, with the following u’s

u u u u u

Here’s how you would do it:

u	u	u	u	u
1	2	3	4	5

The interesting idea is that counting is just a special type of one-to-one matching. Here’s another way of trying to match one counting number with each u:

u	u	u	u	u
1 2	3		4 5	6

If someone doesn’t understand why this way isn’t right, then the person won’t be able to count right. So people have to be able to do one-to-one matching before they can count correctly.

The point of this section is that

A. we count by doing one-to-one matching of numbers with whatever we’re counting,
or
B. counting was one of the most important inventions of the human race?

Our numbering system: A great idea

20. We use 10 types of marks to represent numbers: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. A symbol is something that stands for something else; 0, 1, 2 and so forth are symbols for certain numbers of something. We could, if we wanted to, make up even more symbols for numbers. We could say that after 9 comes A, B, and C. If we made up a new symbol for every number, there would be no end to the

symbols we would have to make up and remember! To be able to represent a million numbers, we would have to have a million different symbols!

A long time ago, somebody did something very smart. That person decided, “Let’s not make a new symbol for each number. Let’s represent some numbers by 2 or more symbols.” For example, suppose we let X represent ten, and I represent one. Then we can let XI represent eleven , XII represent 12 things, and XIII represent thirteen things. We can use combinations of symbols to represent numbers. Using X for ten and I for one was what the Romans did, long ago. They used a variety of other letters to stand for other numbers.

The important discovery this section talks about is

A. using two or more symbols together to represent a number,
or
B. using a symbol to represent zero?

21. The Roman system worked fairly well. But even better is the system we now use, called the Hindu-Arabic system.

Let’s think about the symbol 2, which is also called the numeral 2. Depending on where it is in a number, it can mean 2 ones, 2 tens, 2 hundreds, 2 thousands, and so forth. In the number 27, the numeral 2 means 2 tens. In the number 82, the 2 means 2 ones. How much a numeral stands for depends on what *place* it occupies.

One of the big ideas of the Hindu-Arabic number system is that

A. A symbol, such as 7 stands for a number that depends upon what place it occupies,
or
B. There is no largest number that can be written?

22. Our number system is based on tens. Let’s illustrate why we say this.

Let’s make a picture of, say, 14 things. We’ll pick the letter u as the “thing.”

Here are 14 things:

uuuuu	
uuuuu	uuuu
one batch of 10	four more

If we wanted to, we could represent a batch of ten things by the letter t. So 14 things would be

t	uuuu
one batch of ten	and four more ones.

53 things would be

ttttt	uuu
five batches of ten	and 3 more ones.

What happens when we have 19 things,

t	uuuuuuuuu
one batch of ten	and 9 more ones,

and we get one more one – this one: u?

Now we have

t	uuuuuuuuuu
one batch of ten	and 10 more ones.

But this is the same as

tt
two batches of tens.

It’s as if we batched or bundled up the ten ones, to make another ten. Each time we get more than ten, we make another batch or bundle.

The point of this is that

A. In our number system, each time we get as many as ten things, we make a new bundle of those things,
or
B. if you switch different numerals in a number, you make the number different?

23. Why did someone decide to make a new batch or bundle each time as many as 10 things accumulated? Why didn't that person make a new batch after 4 things, or 6 things, or any other number? It's almost certainly because people have 10 fingers. Thus the number 53 means

all the fingers of 5 people and three more fingers.

This is the same as

ttttt uuu
five batches of ten and three more ones.

The main point of this is that

A. The number 53 has three ones,
or
B. We bundle numbers in groups of 10, because people have 10 fingers?

24. What happens when we get to the number of fingers that 10 people have? We call this a hundred. We bundle those people into a group. We make a new place in our system of numerals, right to the left of the tens' place, for the number of such groups that we have. So the number 324 means

the number of fingers on 3 groups of ten people

plus the number of fingers on 2 more people (and each person is a group of ten fingers)

plus four more fingers.

The main point of this section is that

A. a fraction can be less than one,
or
B. a bundle of ten groups of ten is called a hundred.

25. The higher we go with numbers, the more symbols we need to represent that number. Let's think about the word *digit*. The number 28 has two digits, because it's represented by two symbols, the 2 and the 8. The number 297 has three digits, because it's represented by 3 symbols, the 2, the 9, and the 7.

The point of this section is that

A. there are more three digit number than two digit numbers,
or
B. the number of symbols that it takes to represent a number is called the number of digits in the number.

26. As we go to higher and higher numbers, we define each group as ten of the previous group. A hundred is ten tens. A thousand is ten hundreds. Ten thousand is ten thousands. A hundred thousand is ten ten-thousands. And a million is ten hundred-thousands. We keep following the same rule that we form a new bundle when we get ten of the largest bundle so far. Each time we do that, we make one more "place" in our number system, by adding a digit just to the left of the place before.

The point of this is that

A. subtraction is called the inverse of addition,
or
B. each new bundle in our number system is 10 of the previous bundle.

Expanded form

27. Someone made up the term "expanded form." This means to take each digit in a number and tell how many of which bundle it represents. Here's an example: when we put the number

369
into expanded form, we say
3 hundreds 6 tens and 9 ones.

When we put 1,275 into expanded form, we say
1 thousand 2 hundreds 7 tens and 5 ones.

The point of this section is that

A. the words for some of the numbers aren't as logical as some others,
or
B. expanded form means telling how many bundles (of one, tens, hundreds, etc.) each numeral in a number represents.

28. The Hindu-Arabic system of numbers has something very smart, that the Roman system lacked: a symbol for 0. This symbol comes in very handy when keeping track of how many of each bundle a number represents.

Let's suppose you wanted to represent the number that is three hundreds and nine ones. If you write the number this way

39

then that wouldn't work, because the person reading it would think you meant thirty-nine. So to write three hundreds and nine ones, you use a 0 as a "place-holder." You write 309. The zero makes the 3 go into the place for the number of batches of hundreds, not the number of batches of tens. The numeral 0 is very important for clearing up confusion about whether a digit represents tens, hundreds, thousands, or whatever.

The point of this section is that

A. the numeral 0 is very useful in writing numbers, because it gets rid of confusion about what a certain digit represents,
or
B. it is impossible to divide by the number 0.

29. We use the phrase “place value” to talk about how the value of a digit depends on what place it’s in. It’s good to remember the following sequence: ones, tens, hundreds, thousands, ten thousands, hundred thousands, millions. This tells the order of what the various places represent, going from right to left.

So for the number

2,179,364

the 4 is in the ones’ place
the 6 in the tens’ place
the 3 in the hundreds’ place
the 9 in the thousands’ place
the 7 in the ten thousands’ place
the 1 in the hundred thousands’ place
and the 2 in the millions’ place.

The point of this section is that

A. a million is ten times bigger than a hundred thousand,
or
B. there’s a certain sequence, going from right to left, that tells what the various digits represent.

30. When you read big numbers, it’s useful to divide the digits into groups of 3. Each group of 3 is called a *period*. The first group of 3, going from right to left, is called the ones’ period. The second group of 3 is the thousands’ period. The next is the millions’, and the next is the billions’ period. When you are reading big numbers, you read each group of three, from left to right, and tell the name of the period. When you’ve done that, you’ve read the number. For example,

479,365,102

is divided by commas into three periods. 479 is the millions’ period, 365 is the thousands’, and 102 is the ones’. So when we read this number, we read

four hundred seventy-nine million,
three hundred sixty-five thousand,

one hundred two.

Here’s another example:

17,301,226,498

is divided by commas into 4 periods. 17 is the billions’ period, 301 is the millions’, 226 is the thousands’, and 498 is the ones’. So we read the number

seventeen billion
three hundred one million
two hundred twenty-six thousand
four hundred ninety-eight.

The main purpose of this section is to

A. make sure you remember to put commas in numbers,
or
B. explain how you read large numbers, using “periods”?

Thinking about the words we use for numbers

31. The symbols that we use for numbers are called numerals. The numerals stand for certain numbers of things. The system of numerals that we have to stand for numbers is very sensible. It would be hard to improve on this system. But the words we use for just a few of those numbers are unusual, when you think about them. Let’s think about the numbers between 10 and 20.

The words for 65 and 72 and 89 and so forth make lots of sense. “Sixty-five” means six tens and five ones. The first word (sixty) tells how many tens, and the second word (five) tells how many ones. For 89, the first word (eighty) tells how many tens, and the second word (nine) tells how many ones, just like the first digit (8) tells how many tens, and the second one (9) tells how many ones.

But for “fifteen” and “sixteen” and “seventeen” for example, the first part of the word (the “fif” or “six” or “seven”) tells how many ones, and the second part (the “teen” tells you how many tens. This is backwards from the way it is with the twenties, thirties, forties, and so forth. We just have to remember that "fifteen" means "one ten and five ones."

The words "eleven" and "twelve" are even stranger. They give us no clue at all that they mean "one ten and one" and "one ten and two." We just have to remember what eleven and twelve mean.

But we're lucky that the numbers, when written with numerals rather than words, make perfect of sense. 11 and 12 and 15 and 26 and 89 all mean the number of tens given by the digit on the left and the number of ones given by the digit on the right.

One point this section makes is that

A. whereas words like "fifty seven" name the number of tens first and the number of ones second, words like "seventeen" give the ones and tens in the opposite order,
or
B. the word for "three" begins with the letter t in several languages other than English?

The equals sign

32. This sign = means that two numbers are equal. If one of them is 3, and they are equal, then what's the other number? 3! If one of them is 9, and they are equal, what's the other one? 9.

Here's an idea most people don't think about often: the fact that 2 numbers are equal doesn't mean that the bunch of things that were counted were the same things! Here is a set of exclamation marks:

! ! !

And here is a set of question marks:

? ? ?

It would be correct to say that the

number of exclamation marks above = the number of question marks above

even though the two groups are not of the same thing. The equals sign means that the NUMBER we get when we count them is the same.

An example of the point made by this section would be that

A. There are only two ways for two numbers not to be equal: for the first to be less than the second, or greater than the second,
or
B. the number of pencils in a room could be equal to the number of students in a room, even though pencils and students are not the same thing?

We can let words or letters stand for numbers

33. When we just said that the

Number of exclamation marks above = the number of question marks above,

we were letting words stand for numbers. Suppose that we didn't want to do so much writing. Suppose we said, "Let's call the number of exclamation marks x. And let's call the number of question marks q." Then we could say that the numbers are the same by just saying

$x=q$.

In this example this is just another way of saying that

$3=3$

because there are three exclamation marks and 3 question marks above.

When we make letters stand for numbers, they are called "literal numbers." It turns out that it's very useful to let letters stand for numbers sometimes. One of the reasons it's useful is that we can state rules about how numbers work. These rules will work no matter what numbers we make the letters stand for. We don't have to write millions of different examples. The next section gives an example of one of those rules.

The very useful idea this section states is that

A. 3=3,
or
B. we can let letters stand for numbers?

Equals is "transitive"

34. Suppose someone told you the following. "I have a certain number of dogs. For each dog, there is exactly one collar. (In other words, the number of collars equals the number of dogs.) For each collar, there is exactly one leash. (In other words, the number of leashes equals the number of collars.) Now, what can you say about the number of leashes in comparison to the number of dogs? Are they the same, or different?

If there were 3 dogs, then there would be 3 collars. If there were 3 collars, then there would be 3 leashes. So the number of leashes would be the same as the number of dogs. What if there were 5 dogs? Then there would be 5 collars, and thus 5 leashes. So again the number of dogs and leashes would again be equal.

The conclusion we come to when we think about problems like this is that if a first number equals a second number, and the second number equals a third number, then the first has to equal the third. This rule is called the "transitive" rule.

When we say that equals is "transitive," we mean that

A. if a first number equals a second number, the second has to equal the first,
or
B. If a first number equals a second, and the second equals a third, then the first has to equal the third?

35. Here's how we say that "equals is transitive" in mathematical language, using literal numbers. Let's call the first number a, the second number b, and the third number c.

If $a=b$ and $b=c$, then $a=c$.

Do you notice that we can say the rule in a lot less time when we let letters stand for numbers? One of the main reasons we let letters stand for numbers is that it's easier to do this than to keep talking about "a first number" and "a second number" and so on.

This section makes the point that

A. By letting letters stand for numbers, we can say mathematical ideas in a more streamlined way, using fewer words,
or
B. Some relationships are transitive, but others are not?

36. Let's take a minute to think of a couple more concrete examples of how equals is transitive.

If Jane is exactly as tall as Kara, and Kara is exactly as tall as Lynn, then what can we say about the heights of Jane and Lynn? They would have to be the same, wouldn't they?

Suppose someone gives you a certain amount of money. Then she gives your brother the same amount. Then she gives your mother the same amount that she gave your brother. Did you get the same amount as your mother?

If you can easily think up examples like this, you understand the idea that equals is transitive.

This section is advising that if you want to understand how equals is transitive, it's good to

A. think of concrete examples of this idea,
or
B. write the idea using different letters?

The greater than and less than signs

37. The symbol $<$ means "is less than," and $>$ means "is greater than."

$3<5$ means that 3 is less than 5.

$10>9$ means that 10 is greater than 9.

The small and pointy end of the sign always points to the smaller number. The bigger end of the sign opens up toward the bigger number.

A certain type of problem asks you to draw in a sign to show which number is bigger. For example:

Put in a $<$ or $>$ sign between these numbers to make the statement true.

10 15

Since 10 is less than 15, you put in a less than sign, like this:

$10 < 15$.

This section made the point that

A. If the first number is greater than the second, then one more than the first number has to be greater than one more than the second number,
or
B. the pointy end of the greater than or less than sign always points to the smaller number, and the sign opens up toward the larger number?

Greater than and less than are transitive too

38. Suppose we know two things:

1. I have more socks than I have computers.

2. I have more pieces of paper than I have socks.

What can we figure out from this about whether I have more computers or pieces of paper?

Suppose I have 3 computers. Then, since I have more socks than computers, I would have to have at least 4 socks. And since I have more pieces of paper than socks, I would have to have at least 5 pieces of paper. So I would have to have more pieces of paper than socks. Whatever other numbers we try, it always comes out that I have to have more pieces of paper than computers.

Suppose that I am taller than my wife, and my wife is taller than our friend Lisa. Between Lisa and me, who is taller? I have to be taller, don't I? This is true because "greater than" is transitive.

The main point of this section is that

A. People usually own more pieces of paper than they own computers,
or
B. If one number is greater than a second, and the second is greater than a third, the first has to be greater than the third?

39. One way to solve puzzles about putting numbers in order, especially when they get harder, is to draw lines to show the relation of the numbers.

Here's a sample puzzle. Suppose that Jane is taller than Mickey, and Alex is shorter than Mickey. What is the order of the three people's heights?

Let's do that problem by drawing lines that show how tall people are.

Let's just pick a certain length to represent Jane's height.

Jane: ___________________________________

Since Jane is taller than Mickey, we'll draw a shorter line for Mickey:

Jane: ___________________________________

Mickey: __________________________

and since Alex is shorter than Mickey, we'll draw a line for Alex that's shorter than Mickey's line:

Jane: ____________________________________

Mickey: __________________________

Alex: _________________

We can see from this that the order of heights, from tallest to shortest, is Jane, Mickey, and then Alex.

The point of this section is that

A. Sometimes girls are taller than boys,
or
B. It's useful to draw lines to keep up with the information in puzzles asking you to put numbers in order.

40. When you do enough problems like these, you get very familiar with the rule that says: if a first number is bigger than a second, and a second is bigger than a third, then the first has to be bigger than the third.

Here's another way of saying that. Let's call the three numbers a, b, and c.

If $a>b$ and $b>c$, then $a > c$.

People name this rule the "transitive" rule for "greater than."

This section

A. stated the transitive rule for greater than, using letters as well as words to stand for numbers,
or
B. made the point that “equals” is transitive?

41. Transitivity also applies to “less than.” For example, if the number of teeth I have is less than the number of hairs I have, and the number of eyes I have is less than the number of teeth, then the number of eyes is less than the number of hairs.

If a first number is less than a second, and a second is less than a third, the first has to be less than the third.

Or, if $a<b$ and $b<c$, then $a<c$.

This section

A. taught you how to solve equations by doing the same thing to both sides of the equation,
or
B. stated that “less than” is transitive, using words and letters to stand for numbers?

Greater than or equal to, and less than or equal to

42. There are two more symbols that we need in order to communicate about the relationships of one number to another. Suppose I know that John Smith is at least ten years old: maybe ten, maybe more than ten, but definitely not less than ten. Here’s how we could write this, using a mathematical symbol.

John’s age $\geq$ 10.

The symbol, $\geq$ is read, “greater than or equal to.”

We could also write the following:

10 $\leq$ John’s age.

We would read this as “10 is less than or equal to John’s age.”

The purpose of this section is

A. to tell the meaning of "greater than or equal to" and "less than or equal to" and to tell what signs we use for these,
or
B. to show that "greater than or equal to" and "less than or equal to" are transitive relationships?

The game of mystery number

43. Here's a game that people can play that is really helpful for those who are getting familiar with the number system. One person thinks of a number. At the beginning, you think of a number from 0 to 10. Later you can pick among numbers from 0 to 100 or 0 to 1000.

The second person's job is to guess the number. Each time he guesses wrong, the other person says, "It's greater than that," or "It's less than that." With these clues, the second person gradually homes in on the mystery number. This is a "cooperative" game, because both people are trying for the same goal, that the mystery number is guessed correctly.

Playing this game is a great way to practice thinking about how the number system works and about greater than and less than.

In playing the game, "mystery number,"

A. one person takes guesses, and the other tells whether that guess is too high or too low, until the person gets the right answer,
or
B. one person measures out a certain length, and the other guesses how long it is?

44. The more you play this game, the more you find out that you get to the right answer quickest by taking a guess that's in the middle of the possible answers that are left. For example, let's imagine that the mystery number is in the range from 0 to 100. If on my first guess I say, "Is it 100," and I get the answer, "It's lower than that," I've only eliminated one number from the running. But if on my first guess I say, "Is it 50," no matter whether the answer is "It's higher than that" or "It's

lower than that," I've eliminated about 50 numbers. I'm going toward the right answer much faster.

This section has to do with

A. what the rules of "mystery number" are,
or
B. what the fastest-working strategy for mystery number is?

Subsets are also transitive

45. We've talked about how the transitive rule applies to equals, greater than, and less than. Let's review how equals and greater than are transitive.

If I'm in the same grade as Katie, and Katie is in the same grade as Molly, then what can we say about the grade that Molly and I are in? If you figure out that Molly and I are in the same grade, then you understand how equals is transitive.

If you find out that a dog can run faster than a certain hamster, and a horse can run faster than a dog, then what do you know about which is faster, the horse or the hamster? If you understand that these facts show that the horse is faster than the hamster, then you understand that *greater than* is also transitive.

This section

A. explained how subsets are transitive,
or
B. reviewed how equals and greater than are transitive?

46. What about subsets? Suppose you know that all ducks are birds. You also know that all birds are animals. From this you can conclude that all ducks are animals.

Another way of saying what we just said is as follows. Ducks are a subset of birds. Birds are a subset of animals. Therefore ducks are a subset of animals.

The general rule is that if A is a subset of B, and B is a subset of C, then A is a subset of C. In other words, the relation that we call "is a subset of" is transitive.

Here’s another example. All lizards are reptiles. All reptiles have hearts. Therefore all lizards have hearts. Another way of saying this is that lizards are a subset of reptiles. Reptiles are a subset of animals with hearts. Therefore lizards are a subset of animals with hearts.

The main idea of this section is that
A. if set A is a subset of set B, and set B is a subset of set C, then set A is a subset of set C; in other words, subsets are transitive,
or
B. a set is a subset of another if, and only if, every one of its members is also in the other set?

What type of relation isn’t transitive?

47. The symbol ≠ or <> means “is not equal to.” Let’s look at some examples with that relation. 6≠8, and 8≠6. So if “not equal to” were transitive, then we could say that 6≠6. But 6=6. The transitive rule doesn’t work with the relationship we call “not equal to.”

The point of this is that

A. The equals relationship is transitive
or
B. The “does not equal” relationship is not transitive.

48. What’s another relationship that isn’t transitive? How about “is one more than.” Let’s try it out. 6 is one more than 5. 7 is one more than 6. So 7 is one more than 5? No, it isn’t, so “one more than” is another relationship that isn’t transitive.

The point of this is that

A. “is one greater than” is another relationship that isn’t transitive,
or
B. “greater than” is a relationship that is transitive.

Chapter 2: Addition

Adding when there are "some and some more"

49. A very frequent type of addition problem is called "some and some more." We start with a certain number, and make it a certain amount more. For example, we have 5 dollars, and we get 3 dollars more. To find out how much we have altogether, we add 5 plus 3. For another example, suppose we spend 6 hours working, and then we spend 3 hours more. How many hours have we spent altogether? The answer is 6+3.

Sometimes the "some more" happens in our imagination rather than in real life. Suppose that someone is 7 years old. How old will that person be in 3 years? The person "has" 7 years, and then we imagine that the person "gets" three more years of life. The person would now have 7+3 years or 10 years altogether, and this is the answer to our problem, even though in reality, the person is still 7 years old.

The point of this section is that the idea of addition often involves

A. making things greater,
or
B. some and some more?

Names for the numbers we add and the answer we get

50. We call the two numbers that are added together "addends." We call the number that we get by adding them the "sum." So when we are thinking about 5+3=8, 5 and 3 are the addends, and 8 is the sum.

This section said that

A. The numbers added together are called addends, and the answer to the addition problem is called the sum,
or
B. you can add numbers in any order and still get the same answer?

Chapter 2: Addition

51. One way of doing addition is by putting things together and counting them. Suppose that someone has 3 pennies, and he gets 2 pennies more. How many pennies does he have now? One way that he can find the answer is to throw all the pennies into one pile, and count them. Especially when you get large numbers of things, you want to use easier ways of adding, that don't require you to do so much counting. But it's good to remember that adding is just answering the question, if you take two or more groups and put them together and count them, what do you get.

This section makes the point that addition can be done by

A. starting with the higher number and counting up from there,
or
B. putting the two groups together and counting?

52. Here's another important idea that most people use without ever thinking about: the answer to an addition doesn't depend on what you are adding. It's very useful that this is the case. Suppose you want to know how many lions you have if you start with three and get two more. This would be very difficult to answer if you actually had to go out and find lions to put together into a group and then count! But rather than using lions, we could put up three fingers and then put up two more. The number we get is the same as if you had started with three lions and then had gotten two more. This makes it lots easier, because it's easier to find fingers than to find lions!

The main idea of this is that

A. You get the same answer to an addition problem, no matter what you are adding,
or
B. it's ok to use fingers when you are adding?

53. Even if you already add really well, it's useful to think about different ways of adding. We've already talked about one method: you can put up fingers for the first number, and then put up more fingers for the second number, and then count up how many fingers you have in all.

But when you use this method of adding, you run out of fingers when the answer to the addition problem gets greater than ten.

There is an easier way to add. You can start from the higher number, and just think it, but not put up any fingers for it. You then count up a number of jumps equal to the lower number. You use your fingers to keep track of how many jumps you have made. So suppose I have 25 books and I get 3 more. How many do I have altogether? I say 25 to myself. Then I count up to 26 (and put up one finger) and then 27 (and put up a second finger) and then 28 (and put up a third finger). Now I have 3 fingers up, so I'm done. My answer is 28.

When you use this way of adding, you

A. put up fingers for both addends,
or
B. put up fingers for only one of the addends?

54. When people use this method, sometimes they make mistakes by putting up a finger before they have added anything. Suppose someone is adding 5 plus 3. He thinks, 5, and puts up one finger, 6, and puts up the second finger, and 7, and puts up the third finger. He gets that 5+3 is 7. Where did he go wrong? What he should have done is to think 5, without putting up any fingers because nothing has been added. Then he puts up one finger and says 6, a second finger and says 7, and a third finger and says 8. Now he's done, and he has the right answer. He got the right answer because he put up a finger for each number that he added. Or he put up a finger for each "jump" up that he made.

The idea of this is that

A. you save work if you start with the higher number and count up the number of jumps for the lower number,
or
B. when adding by counting up, you can avoid mistakes if you remember not to start counting until you make the number jump up.

55. When adding by counting up, it's less work if you start with the higher number. But you'll still get the same answer whether you start with the lower number or the higher one. This is because when you are adding two numbers, it

doesn’t matter which one comes first. This is called the commutative law of addition.

uuuu uu

How many u’s are there above? If we look at them starting with the batch on the left, there are 4, and there are 2 more. 4+2 = 6. But nobody said that when we count things, we have to start from the left side. We could count the ones on the right first. We could say there are 2, and 4 more. 2+4=6. The picture shows 2+4, just as it shows 4+2.

Another way of saying the commutative law of addition is that

A. which order the numbers are in doesn’t make a difference when adding,
or
B. it doesn’t make a difference whether the two numbers are equal or different, when adding?

56. Another way of stating the commutative law of addition is by using letters to stand for numbers. Suppose that the letter a stands for some number. And b stands for some other number. If a is 5 and b is 3, then a+b means 5+3. But if a is 4 and b is 6, then a+b means 4+6. Then we can say the commutative law of addition by saying that

a+b=b+a,

no matter what a and b are.

a+b=b+a is another way of saying that

A. the order in which we add numbers doesn’t make a difference,
or
B. a and b can be very large numbers?

57. When we add three numbers, we first add two of them and then the third. For example, in adding

5+2+1

we figure out that 5+2 is 7, and 1 more is 8. We can show that we did 5+2 first by putting parentheses around them:

(5+2)+1.

But what if we had wanted to add the 2+1 first, and then add 5 to that? We would have gotten 3, added that to 5, and gotten 8, just the same as before. We show that we did the 2+1 first by putting parentheses around the two and the one:

5+(2+1).

Whenever we are adding three or more numbers, it doesn't matter which pairs we add first. This fact is called the associative law of addition. It's also called the "grouping doesn't make a difference" law of addition.

The associative law of addition tells us that

A. when we're adding two numbers, order doesn't make a difference,
or
B. when we're adding three or more numbers, it doesn't matter which we group together first?

Parentheses mean to do this operation first

58. Addition is something we can do to numbers. It's called an "operation." Subtracting, multiplying, dividing, and several other processes are also operations. Parentheses, which are these ( ), tell us to do the operation inside the parentheses first.

> (5+2) + 1 means to add 5 to 2 first, getting 7, and then add 1 to the 7.
>
> 5 +(2+1) means to add 2 and 1 first, getting 3, and then add 5 to the 3.

When we have 3 numbers added together, we get the same thing no matter where we put the parentheses – this is what the associative law tells us. But there are other times when parentheses make a lot of difference in what answer we get.

The point of this section is that

A. It's hard to state the associative law without using parentheses,
or
B. Parentheses tell us which operation to do first.

Learning the addition facts

59. Many people who understand a lot of math find math unpleasant partly because they don't know their basic "facts" well enough. When we speak of addition facts, we mean facts such as 8+5=13, 7+8=15, and so forth. All the combinations from 0+0 to 9+9 are what some people would call the basic addition facts; others would also include up to 10+10. Solving more advanced math problems is much more fun and interesting if you can do a lot of the adding or subtracting or multiplying or dividing in your head, quickly. If you have to slow yourself down long enough to use a calculator or do arithmetic on paper, many problems aren't as much fun. Of course, for some problems you must use paper or a calculator, because the arithmetic is too difficult to do in your head. But you'll find a very large number of problems where being able to say the basic facts quickly is really helpful.

The main point of this section is that

A. Sometimes you'll need to use a calculator,
or
B. It's a good idea to get very fast at recalling the basic math facts.

60. Here are some short cuts some people use when learning the basic addition facts.

First, keep in mind that two numbers add up to the same number no matter which order they are in. If you know what 9+1 is, you also know what 1+9 is.

The +0's are easy because adding 0 does not change a number. So 3+0 is 3.

Learning the +1's, like 3+1, 7+1, and so forth, is easy because you just count up one from the number.

The +2's can also be remembered by counting up quickly. What's 5 + 2? You start with 5 and count up 2 by saying 6, 7. So 5+2 is 7.

Learning the "doubles" for some reason seems to be pretty easy for most people. 1+1 =2, 2+2 = 4, 3+3=6, 4+4=8, 5+5=10, 6+6=12, 7+7=14, 8+8=16, 9+9=18, 10+10=20.

So far we have discussed short cuts in learning addition facts. Which short cuts have we discussed so far?

A. One aparts, two aparts, plus 10's, and plus 9's.
or
B. Plus zeroes, plus ones, plus twos, and doubles.

61. If you learn the doubles well enough, you can use them to figure out the sums that are "one aparts" from doubles. These are the sums where the numbers are just one apart from each other, like 3+4, 4+5, and so forth. You can think, for example: 3+3=6. So 3+4=7. Why? 4 is one more than 3, so 7 has to be one more than 6. Here are the one aparts: 1+2=3, 2+3=5, 3+4=7, 4+5=9, 5+6=11, 6+7=13, 7+8=15, 8+9=17, 9+10=19.

You can also use the doubles to figure out sums that are "two apart" from being doubles. These are the sums where the numbers are two apart from each other, like 2+4, 3+5, 4+6, and so forth. With "two aparts," you can take one off the higher number and put it on the lower one, to make a double. So with 4+6, we take one off the 6 (making it 5) and put that one onto the 4, making it 5 too. So 4+6 is the same as 5+5, or 10.

Here are the two aparts: 1+3=4, 2+4=6, 3+5=8, 4+6=10, 5+7=12, 6+8=14, 7+9=16, 8+10=18.

The plus 10's are easy, because you just put a 1 on the left side of the number to get the answer: 10+1=11, 10+2=12, 10+3=13, and so forth.

The plus 9's are easy when you figure that adding 9 is the same as first taking away one and then adding 10. In other words, you take one off the number to make it one less, and put that one on the 9 to make it 10. Then you just add 10. So to add 9+7, you take one off the 7 and put it on the 9, to change 9+7 into 10+6. The answer is 16. Here are the plus 9's: 9+1=10, 9+2=11, 9+3=12, 9+4=13, 9+5=14, 9+6=15, 9+7=16, 9+8=17, 9+9=18, 9+10=19.

After these, there are only 6 more facts to learn. Three of them are plus 3's, that are fairly easy to count up: 6+3=9, 7+3=10, 8+3=11. The last 3 are 7+4=11, 8+4=12, and 8+5=13.

Which short cuts does this section discuss, in learning the addition facts?

A. One aparts, two aparts, plus 10's, and plus 9's.
or
B. Plus zeroes, plus ones, plus twos, and doubles.

62. Here's something else that will help you learn the addition facts. Work to where you can say very automatically the sums that add to 10. 0+10, 1+9, 2+8, 3+7, 4+6, 5+5, 6+4, 7+3, 8+2. 9+1, 10+0. Then sometimes you can think for example like this: What is 8+5? I know that 8+2 is 10. 5 is 2+3. So

8+5 is 8 +2 +3, or 10+3 or 13.

Here's another example:

7+5 is 7+3 +2, or 10+2, or 12.

The point of this section is that

A. It's good to learn very thoroughly the combinations of numbers that add to 10,
or
B. It's good to check your addition each time you do it?

63. However you learn the addition facts, it's good to practice them until you can instantly retrieve them from memory. In a situation where you are writing or typing answers to addition facts, it's good to get to the point where you are doing over 30 facts per minute, or one fact in less than 2 seconds. If you are looking at addition problems and saying the answers, it's good to be able to say at least one fact per second, or 60 per minute.

This section recommends that

A. you practice addition facts a certain amount per day,

or
B. you practice addition facts at least until you can answer them fast?

When you are adding measures, they have to be in the same unit

64. Here's a "some and some more" problem, involving adding. A tree is 3 feet tall. The tree grows 6 inches. How tall is it now? Suppose someone says, "That's a some and some more problem, so I just add. 3+6=9. So the answer is 9." This person has gotten the wrong answer. The answer is wrong because when you add measures, you have to use the same units. Three feet is the same as 36 inches. If we add 36 inches and 6 inches, we get a meaningful answer. The tree is now 42 inches tall. Or, we could say, 6 inches is one half a foot. Three feet plus one half a foot is 3 1/2 feet. As long as the measures are in the same unit, we get a meaningful answer.

Another example of the point of this section is that

A. if someone works 3 hours one day and 82 minutes the next day, and we want to know how much the person has worked in all, we can't just add 3 and 82; we have to get the numbers into the same unit,
or
B. units like centimeters and meters involve easier calculations than units like feet and inches?

65. There's an old saying that "You can't add apples and oranges." Let's think about that a little more. Suppose someone asks you, "You have 2 apples and 3 oranges. How many do you have altogether?" You have to respond to this question with another question: "How many what?" If the question is, how many fruits, you add 2+3 and get 5, you get the right answer to this question. So in this case you can add apples and oranges! But what you're really doing is adding 2 fruits plus 3 fruits and getting 5 fruits. You can think of this as converting them to the "same unit" before adding. On the other hand, if the person says, "You have 2 apples and 3 oranges; how many apples do you have," then the answer is 2. You add 2 apples and 0 apples more. Finally, if the question is, "How many Macintosh apples do you have," and both your apples are Granny Smith apples, then the answer is zero!

This section, as the last one, gives examples of the fact that

A. when you are given two quantities and asked “How many altogether,” you don’t just add without thinking; you have to think about how many of WHAT you are dealing with,
or
B. adding is the opposite of subtracting?

We find perimeters by adding

66. Suppose you take a hike, in a triangular (three sided) path. The first side is 2 kilometers; the second side is 3 kilometers; and the third side, which is 4 kilometers, brings you back to where you have started. How far have you gone in all? This is a pretty straightforward “some and some more” addition problem, isn’t it? You walk 2 kilometers, then 3 more, then 4 more, for a total of 2+3+4 or 9 kilometers.

The distance around any triangle or any other path made of straight lines is called the *perimeter* of the triangle or whatever other figure we are going around. The word perimeter comes from two parts: peri means around, and meter means measure. So the perimeter of something is the measure of how far it is to go around it.

You can find the perimeter of a figure just by adding up the lengths of all the sides, no matter how many sides there are.

The main point of this section is that

A. “peri” means “around,”
or
B. you find the perimeter, or distance around, a figure by adding up the lengths of all the sides?

Chapter 3: Subtraction

The Idea of Subtraction

67. There are lots of times when we start with a certain number, but then instead of adding more, we take some away. For example, we have 5 dollars, but we spend 3 of them. We have 10 stamps, but we use 3 of them to send letters. There are 6 oranges, but we eat 3 of them. In all these examples there is some sort of "taking away." We do something to make the first number smaller rather than bigger. This is what's behind the idea of subtraction.

We write "5 take away 2, leaves 3" by saying "5 minus 2 equals 3" or 5-2=3.

Subtraction is often thought of as the opposite of addition, because instead of having "some more" you are having "some fewer" or "some less."

The purpose of this section is to

A. tell you how to remember subtraction facts,
or
B. tell the meaning of subtraction.

Words for the numbers in a subtraction problem

68. In a subtraction problem, the number that we start with is called the minuend. The number we take away is called the subtrahend. The answer is called the difference. So when we talk about 5-3=2, 5 is the minuend, 3 is the subtrahend, and 2 is the difference.

A summary of this section is that

A. minuend minus subtrahend equals difference,
or
B. numbers that are added or subtracted are called "terms"?

Subtracting with fingers

69. We talked about two ways of adding with fingers. There are two similar ways of subtracting with fingers.

Suppose there were 7 dogs, and we gave 2 of them away. How many are left? We can pretend that each of 7 fingers is a dog, and put up 7 fingers. Then we put down two fingers, one for each dog that was given away. Now we count the number of fingers that are still up. The answer is 7-2, or 5.

The problem with this way of subtracting is that it's hard to use when the number you start with is over 10. But here's a second way to subtract 7-2. You just think the number 7, and don't put any fingers up. You count backwards, and you put up a finger each time you take a jump backwards. You stop when there are two fingers up, and the number you have just said is your answer. So you start with 7. You say "6," and put one finger up. You say "5," and put another finger up. Now you've got two fingers up, so you're done. So 7-2 is 5.

This section told you

A. two separate ways of subtracting using fingers,
or
B. why you should feel OK about using your fingers to subtract?

Subtraction by counting up rather than counting down

70. There's another way of subtracting that lets you count forward rather than backward. What's 8-6? We could count backwards from 8 for 6 jumps, putting up one finger each time to keep track of our jumps. We would go 7, 6, 5, 4, 3, 2. Since we land on 2, the answer is 2.

But we could do this problem in a different way that is easier. We count how many jumps upward from 6 we have to take to get 8, and the answer will be the difference between 8 and 6. We first think "6." Then we put up 1 figure as we count "7" and another finger as we count "8." Since we've landed on 8, we're done, and since we're holding up two fingers, the answer is 2.

In the counting up method of subtraction, the answer is the number of fingers we are holding up when we land on the bigger number. In the counting down method, the answer was the number we landed on when we held up the number of fingers that was taken away.

Even if you're already very good at subtracting, it will probably stretch your thinking a little bit to make sure you fully understand these two methods.

This section focused on the method where you

A. count backwards from the minuend
or
B. count forward from the subtrahend?

The number Line

71. We can imagine, or draw, a line with numbers along it in order. Each number is represented by one point on this line. Because numbers don't ever end, the line could go on forever. Later, we'll talk about how the number line can go on forever in both directions, left and right. But the piece of the number line we'll look at now goes from 0 to 21. The point on the line that's directly above any number represents that number.

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21...

The number line draws above stops at 21 because

A. There's something special about the number 21
or
B. That was about how many numbers fit across this page?

Adding and subtracting using the number line

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21...

72. Let's illustrate using the number line to add, with a simple problem. Suppose we have 4 pet fish, and we get 3 more. How many do we have now? We can add using the number line. We put our pencil eraser or fingertip on the 4. That's how many fish we start out with. Then, to represent that we get three more, we make

three jumps to the right, moving one number higher each jump. We count 1, 2, 3 as we make those jumps. After 3 jumps, we land on 7, so 4+3 is 7.

How do you add with a number line?

A. You start with one of the numbers and make a number of jumps to the right that's equal to the other number. The number you land on is the answer.
or
B. You start at zero, and jump a number of steps equal to the first number. Then you go back to zero, and make a number of jumps equal to the second. The distance between the two landing places is your answer.

---

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21...

73. Now let's subtract with the number line. Suppose we have 7 potted plants, but we give 2 away. How many do we have left?

To do this using the number line, we start at 7. We make 2 jumps. But because we are giving plants away rather than getting new plants, we want to jump in the direction of lower numbers, not higher numbers, to get our answer. So we start at 7 and make 2 jumps to the left. We count "1, 2" as we make those jumps. We land on 5, so 7-2 is 5.

When you start at a certain place on the number line, and make 2 jumps to the left, you are

A. adding 2 to the number you started with,
or
B. subtracting 2 from the number you started with?

Addition and subtraction fact families

74. Let's look at the following batch of u's:

uu uuu

We have a group of 2, and a group of 3, and together they make 5. From this we can see that 2+3 =5. But if we go from right to left, we also see that 3+2 =5.

Now if we think about all 5 u's, and put a finger over the group of 3, we see that 2 are left: 5-3=2. If we think about all 5 and put a finger over the group of 2, we see that 3 are left: 5-2=3. So this drawing can illustrate four facts in all:

2+3=5
3+2=5
5-2=3
5-3=2

These four facts are called a "fact family" for addition and subtraction. Any addition or subtraction fact has three other "relatives" in the same family.

If we know that

123+456=579,

then we also know that

456+123=579
579-456=123, and
579-123=456.

A way of summarizing this section is that

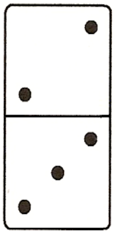
A. you should be careful to watch the sign when you are given an addition or subtraction problem,
or
B. if you know any addition or subtraction fact, you can also figure out the three other facts that are in the same family?

Getting fluent in addition and subtraction fact families by using dominos

75. If you have to stop and think, or count, each time you add 8+6 or subtract 16-9, you will find math much more tedious than you will if you are very fluent in addition and subtraction. The word *fluent* means that you can rattle off these facts automatically, without having to think much. A major way of testing fluency is to see if you can say the facts very fast.

One good way of getting fluent in addition and subtraction is by using dominos. Dominos allow you to get a visual image of numbers. If you forget one of the facts, you can always count the dots on the dominos to check your memory.

Here’s what you do. Look at this domino:



When you see this one, you first notice that there are 2 dots on the top panel, and 3 dots on the bottom panel. There are 5 dots in all. So this illustrates two addition facts: 2+3=5, and 3+2=5.

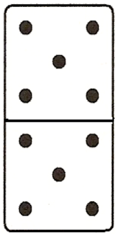
Suppose you were to look at all 5 dots, and then cover up the 2 on the top panel. You would see 3 left. Or, if you were to look at all 5, and cover up the 3 on the bottom panel, you would see 2 left. So this domino also illustrates two subtraction facts: 5-2=3, and 5-3=2.

The goal you want to shoot for is that when you see any domino in a double-nine set, you can rattle off the four addition and subtraction facts it illustrates, very fast.

The purpose of this section was to

A. show that 3+2=5,
or
B. to explain how to drill on addition and subtraction facts, using dominos?

76. Most dominos will illustrate four facts: two addition and two subtraction. But ten dominos in the double nine set illustrate only two facts. They are the “doubles,” the dominos where both panels have the same number of dots. When you look at the following domino



you would just say, “5+5=10, 10-5=5.”

The dominos illustrate

A. two facts each,
or
B. two facts each for doubles, and four facts each for all the rest?

77. At the back of this book there are some pictures of dominos, in case you don’t want to buy a set of double nine dominos. There is a number over each domino. I recommend timing yourself, first with a small group of them, and then working your way up to all 55. When you can do them very fast in order, have someone call out the numbers of the dominos in a random order, and see if you can say the facts for each domino very fast, out of order.

This section recommended

A. specific speed goals you should shoot for,
or
B. a strategy for practicing addition and subtraction facts using the pictures of dominos in this book?

Chapter 4: Regrouping in Addition and Subtraction

Adding two digit numbers by doing the ones and the tens separately

78. Suppose you want to add 25 and 42. One way to do it would be to get a whole bunch of beans or pennies or anything else we could use as a counter. We might get 25 beans, and then get 42 more beans, and then count up how many we have altogether. This would be a lot of work.

One of the wonderful things about our number system is that we don't have to do all that work. We can add the tens and ones separately.

Let's represent tens by the letter t and ones by the letter u.

25 would be represented like this:

t t u u u u u

and 42 would be represented like this:

t t t t u u.

Now let's put them together, and see how much we have altogether.

t t u u u u u (this top line is 25)
t t t t u u (this second line is 42 more)

All we have to do is to count up the ones, and count up the tens separately. There are 7 u's and 6 t's. 6 tens and 7 ones make 67.

The main point of this section is that

A. any time we add two numbers, we could get a number of objects equal to each number, throw them all together, and count them, to get the correct answer,
or

B. when we add numbers with tens and ones, we can reduce our work by adding the tens and the ones separately?

79. An easier way to do this is to write one number over the other, and just add up the ones and tens separately.

$$\begin{array}{r} 25 \\ +\underline{42} \\ 67 \end{array}$$

It’s good to add the ones first and then the tens, rather than going from left to right. We’ll talk about why this is, very soon.

This section gives an example of the fact that

A. when we are doing ordinary addition of two digit numbers, we are just adding the number of ones, and the number of tens, separately,
or
B. we can think of 42 as three tens and twelve ones, just as we can think of it as four tens and two ones?

Regrouping when there are 10 or more in a certain place

80. Let’s make another picture of a number, and decide what number it is.

t uuuuu
uuuuu u

On the left we have a t, that stand for ten. On the right we have 11 more u’s. So we have one ten and eleven ones. If we start counting at ten, and count up one more for each u that we see, we’ll get to 21. So our picture represents 21.

Let’s take a batch of ten u’s out of that group of 11 on the right, and bundle it together to make a ten. Now our picture looks like this:

tt u

This picture makes it more clear that we have two tens and one one. This is more clearly the number we call 21.

Whether we call the number “one ten and eleven ones” or “two tens and one one,” or “21,” the number is the same. Taking batches of ten and moving them from one column to another is called regrouping.

The point of this section was that

A. you’re not allowed to have more than ten ones in the representation of a number,
or
B. you can represent the same number in different ways, depending on whether a certain batch of ones is bundled together to make a ten, or unbundled and left as ones?

81. Suppose we are given a problem like this: Write a two-digit number that is equal to 4 tens and 14 ones.

In our standard two-digit numbers, we bundle all the batches of ten together that we possibly can, so that there are no more than ten ones. So to get the answer, we need to get all the tens we can over into the ten’s column, and leave less than ten ones for the ones column.

Here’s a picture of 4 tens and 14 ones. Let’s represent each of the tens by a little t, and each of the ones by a little u.

```
t t t t          uuuuu
                 uuuuu   uuuu
```

Let’s take ten of the ones and turn them into a ten. Now we have

```
t t t t t       uuuu
```

Now we have 5 tens and 4 ones. This is the form where we can write a two digit number. 5 tens and 4 ones = 54.

This section gave an example of

A. how you can change a certain number of tens, and a certain number of ones that is more than ten, into a standard number,
or
B. how you can subtract using regrouping or “borrowing”?

82. We don’t even have to make a picture to do problems like this, where we change a certain number of tens, plus a number of ones greater than ten, into a standard number. We can just take a ten and put it in the tens column, in our minds. Here’s an example.

Problem: Write a two-digit number for 3 tens and 12 ones.

We think: 12 ones means one ten and 2 ones. Let’s leave the 2 ones in the ones’ column, and move the ten over to the tens’ column. When we add that one ten to the three tens we’ve already got, that gives us 4 tens. So we have 4 tens, and 2 ones left over. That is 42.

This section

A. explained that multiplying is the same as repeated addition,
or
B. gave another example of grouping ten ones to make another ten?

83. Just as we can turn a number with ten or more ones into a standard number, we can also turn a standard number into a number with ten or more ones. Suppose we have this problem:

Express the number 83 as a certain number of tens, and ten or more ones.

We think: 83 means 8 tens and 3 ones. We can “unbundle” one of those 8 tens, to make ten more ones. That will leave us with 7 tens, and 13 ones. And that’s the answer to our problem.

The general point this section is trying to make is that

A. 83 is equal to 7 tens and 13 ones,
or
B. You can express numbers in different forms by “unbundling” tens to make more ones, just as you can “bundle” ones to make more tens?

84. The examples so far have just involved one bundle of ones. But sometimes we can move two or three or more bundles from the ones’ group to the tens’ group, or vice versa. Suppose we have this problem:

What standard number represents 6 tens and 23 ones?

We think: 23 ones is the same as 2 tens and 3 ones. So we have a total of 6 tens plus 2 tens, or 8 tens, and 3 more ones. So the standard number is 83.

This section makes the point that

A. You can bundle or unbundle two or more batches of tens, not just one,
or
B. bundling is useful for addition, and unbundling is useful for subtraction?

85. The examples we have looked at so far involve bundling ones and unbundling tens. But ten tens make a hundred, ten hundreds make a thousand, and so on. We can bundle and unbundle those numbers too. Suppose we have the following problem:

Express the number 4 hundreds, 13 tens, and 0 ones as a standard number.

We think: 13 tens is ten tens and three more tens, or one hundred and 3 tens. So we have 5 hundreds in all, 3 tens, and 0 ones. So the standard number is 530.

Now let’s think about the opposite kind of problem:

Express the number 763 in a form with 10 or more tens.

We think: If we want more tens, we can get ten more of them by unbundling one of the hundreds. So when we change one of those 7 hundreds to ten more tens, we have 6 hundreds, 16 tens, and 3 ones. And this is the solution to our problem!

This section illustrates that

A. we can do bundling and unbundling by changing tens to hundreds and hundreds to tens, not just by changing ones to tens and tens to ones,
or
B. there are ten thousands in ten thousand?

86. Why are these processes of bundling and unbundling so important? Because when we add, we will frequently wind up with more than ten in a certain place, that we'll need to bundle up and shift over to the next place. This is called regrouping or "carrying." Also, when we subtract, we'll frequently need to unbundle a number to make enough to subtract from; this is called "borrowing."

This section makes the point that we are interested in the process of bundling and unbundling numbers because

A. it helps us understand how our number system works,
or
B. bundling is frequently necessary in addition, and unbundling is frequently necessary in subtraction?

Regrouping when adding

87. Bundling and unbundling, or regrouping, is useful when we do calculations.

Suppose we add 45 and 27.

Let's represent the tens and ones with t's and u's, as we did before.

45 is represented like this:

t t t t   uuuuu

and 27 is represented like this:

t t uuuuuuu.

Now let's put them together.

t t t t uuuuu
t t uuuuuuu

Looks like we have 6 tens and 12 ones. Let's take 10 of the ones, and turn them into a ten. We would have this:

t t t t uu
t t t

So 6 tens and 12 ones is the same as 7 tens and two ones. What we've done is just like the regrouping problems we did earlier. So our answer is 72.

This section gave an example of how

A. In adding, we came out with more than ten ones, so we bundled ten of them and added that ten to the number of tens,
or
B. In adding, sometimes it is not necessary to regroup at all?

88. Here's an easier way to regroup when we are adding:

First we write the numbers we're going to add, over each other.

$$\begin{array}{r} 45 \\ +\underline{27} \end{array}$$

Then we start with the ones' column. We add 5 and 7, and get 12. That's the same as two ones, and one ten. We can put down the 2 in the ones' column, and "carry" the ten over to the tens' column. To remind us that we have an extra ten, we write 1 over the 4.

$$\begin{array}{r} 1 \\ 45 \\ +\underline{27} \\ 2 \end{array}$$

Now we add up the number of tens. We get a total of 7 tens: 4 + 2 gives us six, and the 1 more we "carried" from the ones column gives us 7.

$$\begin{array}{r} 1 \\ 45 \\ +\underline{27} \\ 72 \end{array}$$

We get 72, the same thing we got earlier by making a picture of the ones and tens.

When we add the ones in a problem like this, we might end up with 10 or more. If we do, we have to regroup or "carry" a ten over to the tens's column. This is why we start adding with the ones' column rather than starting on the left.

The point of this section is that

A. when we "put down the ones and carry the tens," we are simply bundling up one or more group of tens and adding that group to the tens we already have,
or
B. why we "carry" when adding is very difficult to explain?

Subtracting two digit numbers

89. We discovered that in adding two digit numbers, we don't have to do large amounts of counting – we add the ones and the tens separately. We can do the same thing when we are subtracting.

Suppose we are subtracting 21 from 43.

We can represent 43 like this:

t t t t   u u u

because it has 4 tens and 3 ones.

Now let's take away 2 tens and 1 one. We'll draw a line through 2 tens and one one.

t t ~~t~~ ~~t~~   u u ~~u~~

How much is left without a line through it? The answer is 2 tens and 2 ones. 2 tens and 2 ones is 22. So 43 minus 21 is 22.

An easier way to do this is to write the minuend over the subtrahend, and subtract first the ones, and then the tens.

Here's the first step. 3 ones minus 1 one is 2 ones.

$$\begin{array}{r} 43 \\ -\underline{21} \\ 2 \end{array}$$

Here's the second step. 4 tens minus 2 tens is 2 tens.

$$\begin{array}{r} 43 \\ -\underline{21} \\ 22 \end{array}$$

So the answer is 22.

The problem we did in this section

A. forced us to unbundle a group of ten to make more than ten ones, so we could subtract,
or
B. did not cause us to do any bundling or unbundling?

Regrouping to make ten or more in the ones' place

90. Sometimes when we subtract, we need to "unbundle" a ten, to make ten or more in the ones' place on purpose.

Suppose our number is 31. We want to "unbundle" one of the tens. Let's represent 31 by three t's and one u, like this:

t t t u

If we unbundle one of the tens, that gives us ten more ones. What we have looks like this:

t t uuuuu
uuuuu
u

So instead of 3 tens and 1 one, we have 2 tens and 11 ones. This is still a way of representing 31 things.

We can do this without making a picture of the tens and ones. We just think:

Start with 3 tens and 1 one. Take apart one of the tens. That leaves 2 tens. When we add the ten ones to the one one we already have, that gives us 11 ones. So 3 tens and 1 one is the same as 2 tens and 11 ones.

Here are some other examples:

4 tens 3 ones = 3 tens 13 ones

2 tens 5 ones = 1 ten 15 ones

4 tens 8 ones = 3 tens 18 ones

This section gave examples in which

A. when you have ten or more ones, you bundle one or more groups of ten of them together, to make a standard number,
or
B. you can unbundle a ten to make a number with ten or more ones?

Regrouping when subtracting

91. Suppose you are subtracting 14 from 33. Let's represent 33 like this:

t t t u u u

Now to subtract 14, we want to strike out 1 ten and 4 ones. But we only have 3 ones! What do we do?

We can unbundle one of the three tens, and make it into ten ones. This leaves us with 2 tens and 13 ones, like this:

t t u u u u u
u u u u u
u u u

2 tens and 13 ones is still a way of representing 33.

Now we've got plenty of ones. To subtract 14, we strike out one ten and 4 ones.

t ~~t~~ ~~u u u u~~ u
u u u u u
u u u

How many are left without lines through them? 1 ten and 9 ones. So 33 minus 14 is 19.

We can do the same thing with numbers, without picturing the tens and ones. We first write the problem:

$$\begin{array}{r} 33 \\ -\ \underline{14} \end{array}$$

We think, “We need more ones if we want to subtract 4.” So we unbundle one of the tens in 33, to make 33 into 2 tens and 13 ones. We represent that like this:

$$\begin{array}{r} 2\ 13 \\ \cancel{33} \\ -\underline{14} \end{array}$$

Now that we have enough ones, we can subtract as we have before. 13 minus 4 is 9, and 2 minus 1 is 1. Our answer is 19.

$$\begin{array}{r} 2\ 13 \\ \cancel{33} \\ -\underline{14} \\ 19 \end{array}$$

This section explained the process of

A. unbundling a ten to make ten or more ones, when subtracting,
or
B. getting negative numbers when you subtract a bigger number from a smaller one?

Adding or subtracting numbers of differing lengths

92. Suppose someone had this addition problem: 1,200 +79 +6. And suppose the person set it up and added it like this:

$$\begin{array}{l} \ \ 1200 \\ \ \ 79 \\ +\underline{6\quad\ \ } \\ 15100 \end{array}$$

Reading about Math

With any sort of math problem, it’s good to be able to estimate the answer and see whether the answer we got even seems close to the right answer. If we estimate, we figure that 79+6 is somewhere between 80 and 90. And even if we added 100 to 1200, we’d only get to 1300. So an answer that’s over 15,000 is really far off, isn’t it?

When we add, it’s important that we add digits that have the same place value. We add ones to ones, tens to tens, hundreds to hundreds, and so forth. But the way the problem was set up above, the person added the 1 thousand in 1200 to 7 tens in 79, and to 6 ones in 6, mixing up what was added to what. That’s why the answer came out so far off.

How should the problem have been set up? We have to line up the ones under the ones, the tens under the tens, and so forth. With whole numbers (all of which have a ones’ digit) if we just line up the ones’ digits in a straight column and keep all the other columns neat, we’ve got it made. Here’s how the above problem should have looked:

$$\begin{array}{rr} & 1200 \\ & 79 \\ + & \underline{\quad 6} \end{array}$$

When we add this way, we get 1285, which strikes us as just about right!

What we said about lining up numbers with ones under ones, tens under tens, and so forth, goes exactly the same for subtraction as it does for addition!

The main point of this section is

A. when adding or subtracting, it’s important to watch the sign carefully, so you won’t be adding when you should be subtracting or vice versa,

or

B. when adding or subtracting, you line up numbers so that the digits of the same place value are in a neat column?

Chapter 5: Word Problems, Part 1. Addition and Subtraction

93. Why does math exist? Because people constantly run into questions about their lives that can only be answered by math. If you describe one of these life problems in words so that someone else can work on it, you've written a "word problem." Word problems are the type of math puzzles that people need to figure out for some reason other than that they are assigned by a teacher!

Here's an example of one of the simplest sorts of word problems.

Matthew has 50 dollars saved up. He gets paid 10 more dollars for taking care of a neighbor's pet. How much money does he have now, in all?

It's easy to imagine why Matthew might wonder how much money he has saved, other than that this question might be on a school test. Perhaps he wants to buy something that costs $58, and he wonders whether he has enough yet. He could lay out all his money and count it up, but it's much easier for him to use addition.

The point of this section is that

A. you should learn to do word problems because they are often on tests,
or
B. word problems are the types of problems life presents us, that we do for reasons other than being in school?

94. There are word problems that use math from the simplest addition problem up to very advanced types of math. For every step along the way, there are word problems describing the real-life questions that can be answered using math.

If you are an engineer in charge of building a bridge that will be sturdy, you have many math word problems to solve, and people's lives depend upon your solving them correctly.

If you are a doctor running a research study to see whether a certain drug helps or hurts people, you have many word problems to solve, and again, people's lives may depend upon your solving them correctly.

If you have a home improvement construction business, each time a customer calls you to ask you to do a job, you have a word problem on your hands: “Given these facts about the job, how much should I tell the customer it will cost?”

The overall general point made by this section is that

A. when people have a party, and figure out how much food to buy, they are doing word problems,
or
B. for just about every level of complexity of math, there are real-life word problems that can be answered by that type of math?

95. For all word problems, no matter how simple or complicated, you have certain facts that you know, (usually numbers) and a certain conclusion from those facts (usually another number) that you want to find. In the problem about Matthew earlier, the $50 he had saved and the $10 he earned were the “knowns” or what we were “given,” and the total he had after getting paid was the “unknown” or what we were asked “to find.”

In solving a word problem, you have to figure out what “operations” to do on the numbers that are known, to get the numbers you want to find. In other words, you have to decide whether to add, subtract, multiply, divide, or whatever. In the case of the problem we looked at, we have to take the two “givens,” $50 and $10, and add them, to get $60.

A summary of the main points of this section is that

A. In word problems you have knowns and unknowns. You have to figure out what operations to do on the knowns to get the unknowns.
or
B. If you have $50 and make $10 more, you then have $60.

96. How do you figure out what operations to do, when the problem doesn’t tell you whether to add, subtract, multiply, or divide? One answer is that you need to read the words in the word problem very carefully. Often it helps to make a picture in your mind, or to actually draw a picture, of what the words are telling you. It helps a lot if you’ve seen and worked similar sorts of problems before! But there’s a certain magic in it, too – a moment when a “light bulb” gets lit in your mind, a moment when the answer comes to you in a way that you can’t quite

explain. If you can cultivate a real sense of pleasure from having those "light bulb" experiences, you will find that math offers a never-ending source of enjoyment.

A summary of this section is that figuring out what operations to do in word problems comes from

A. memorizing a list of phrases that clue you what operation to do,
or
B. careful reading, making pictures, using past experience, and having a "light bulb" experience that has a sort of magical quality?

97. Most addition problems can be summarized by the phrase, "some, and some more."

You have 3 pencils, and you get 4 more. How many do you have altogether?

There are 3 power plants in the state, and 4 more are built. How many are there now?

A plant is 3 feet tall, and it grows 4 feet more. How many feet tall is it now?

A used bookstore owner buys a book for 3 dollars, and decides to charge 4 dollars more than that to the customer. How much does the owner charge the customer?

Someone earns $3 on Monday and $4 on Tuesday. How much has the person earned in the two days?

All of these problems are examples of the idea of "some, and some more," and in all of them, you add 3+4 to get the answer.

This section gave examples of

A. different operations you can use in solving a word problem,
or
B. different word problems that are all examples of "some and some more," and all solved by the same operation?

98. There are certain phrases that give you clues about whether to add or subtract. When you're asked to find "the total" or "how much (or many) altogether" or

"how much (or many) in all," you've gotten a clue that it's time to add. When you're asked to find "how many are left" or "how much more" one is than another, or "how much less" one is than another, or "how many more are needed," or "how many are not" a certain thing, you've gotten a clue that it's time to subtract.

The main point of this section is that

A. Whenever you see a question about "How many are left," you know for sure that you need to subtract.
or
B. Certain phrases give you clues about whether to add or subtract in a word problem?

99. You can't figure out what operation to do by just memorizing a bunch of phrases and looking for them in word problems. Here are some examples of this.

Someone has memorized that "how many altogether" means "add." Then the person gets the following problem:

There are 3 boxes. There are 4 pears in each box. How many pears are there altogether?

If the person adds 3 and 4 to get 7, he has used the adding clue in the wrong way. There are 4 in the first box, 4 in the second, and 4 in the third. Drawing a picture might help him see this. He has to add 4 +4+4 or multiply 4 x 3 to get the right answer.

This section gave an example of a situation where

A. "how many altogether" did not mean to add the two "knowns" you were given,
or
B. "how many less" did not mean to subtract?

100. "How many more" is often a clue to subtract. Here's a problem where that phrase would give you the correct clue.

John has earned \$15. Ted has earned \$7. How many more dollars has John earned than Ted?

We can find this out by subtracting 15-7. John has earned 8 dollars more.

However, consider the following problem, which also has "How many more" in it.

John is $8 richer than Ted. Then John earns $3, while Ted is watching television. John now has how many more dollars than Ted?

Suppose that someone thought, "*How many more* means to subtract. So 8-3 equals 5. The answer is 5." The answer would be wrong.

This section gives an example of how

A. "How many in all" doesn't always mean you should add,
or
B. "How many more" doesn't always mean you should subtract?

101. It's always good, when you have solved a word problem, to look at your answer and think, "Does this make sense?"

Let's do this for the problem mentioned in the previous section. Someone subtracts and gets that John has $5 more than Ted. But then we ask whether the answer makes sense. We think, "Hmm. John started out $8 richer than Ted. Then John earned money, while Ted earned none. And now John's lead over Ted has decreased? That doesn't make sense. John's lead over Ted should increase!"

By asking this question, we would figure out that John's lead is first $8, then it is increased by $3, and the correct answer is that John now leads Ted by $11. We had to add instead of subtract, even though the problem asked us "How many more."

A point made by this section is that

A. You subtract when you want to move to the left on a number line,
or
B. You should picture what is going on in the real-life situation the problem asks about, and ask yourself whether the answer you got makes sense?

Four types of subtraction problems

102. It's useful to think of four types of problems that you use subtraction to find. The first is called "taking away." Here's an example of taking away. There are 5 people at home. 2 of them leave. How many are left at home? We call this a

"taking away" problem because two of the five people were "taken away" from the house.

The second type of problem is called a "difference" problem. Here's an example. Sara is 5 feet tall. Jean is 3 feet tall. How much taller is Sara than Jean? Sara is 2 feet taller. You can find the answer by subtracting 3 from 5: 5-3=2. But you aren't really "taking away" feet from Sara. You're asking how much bigger one number is than another. Subtraction tells you the answer.

Richard gets paid $20 per hour, and Jed gets paid $7 an hour. What kind of problem is it if someone asks, "How much more per hour does Richard get paid than Jed?"

A. A "taking away" problem
or
B. A "difference" problem

103. Another type of subtraction problem that is very similar to a difference problem is called a "missing addend" problem. In a missing addend problem, two numbers are added together to get the sum, which is the biggest number of the three. In a missing addend problem, you are given the sum and one of the numbers that's added to get it; you have to find the other. Here's an example. Tom had 5 dollars. He earned a certain amount of money today. After that, he had 9 dollars. How much money did he earn, today? 5 dollars plus a certain missing addend equals 9. We find that missing addend by subtracting 5 from 9. We get 4. Now we can check our answer. If you start with 5 dollars and add 4, you do get 9.

We can solve missing addend problems by just trying out different numbers. But it's easier to do it by subtracting, especially when the numbers get bigger. If the problem is, "A certain number added to 4 gives 9," we can just try a few, or remember that 4+5=9. But if the problem is, "A certain number added to 23 gives 57," it's lots easier to subtract instead of guessing and checking.

Mary was 36 inches tall. She grew a certain number of inches, so as to become 48 inches tall. If we want to know how many inches she grew, this would be a

A. "taking away" problem
or

B. a “missing addend” problem?

104. A fourth type of subtraction problem is called “movement along a number line.” Suppose that the temperature starts out at 21 degrees, and then it gets 5 degrees colder. Can you figure out how cold it is after that?

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21...

Imagine that our number line is a thermometer, held sideways. We start out at 21 degrees. Then the reading on our thermometer moves 5 degrees in the direction of smaller numbers. If we make 5 jumps to the left from 21, what number do we land on? It’s 16, and 21-5=16. It is 16 degrees after getting cooler.

Here’s another example of a “movement along a number line” problem that uses subtraction. Suppose we are 21 miles from home. We come 5 miles closer to home. Now how far away from home are we? Let’s imagine that our home sits at the “0” point on the number line, and we start out on the 21. Then let’s imagine that we make 5 jumps closer to the 0. We land on the 16, again. 21-5=16, and we are 16 miles away from home.

A person has 5 telephones, but gives 4 of them away. If we want to know how many this person has left, this would be what kind of problem?

A. A “taking away” problem,
or
B. A “movement along the number line” problem?

Introduction to multi-step word problems

105. Sometimes when we do word problems, we need more than one step. We need to figure out the answer to an “intermediate question” before we can answer what the problem is asking us. Here’s an example of an intermediate question.

The problem is: Someone bought one thing for $3, and another for $4. How much change does the person get from a $10 bill?

What's the intermediate question? What do we need to know before we can find out the answer? We need to figure out how much total the person spent. When we know that, we can subtract it from $10 to get the change.

So we answer that intermediate question by adding $3 and $4, to get $7 as the total amount spent. Once we have that, we can subtract 7 from 10 to get $3 as the amount of change he should receive.

Which was the "intermediate question" in the above example?

A. How much money did he receive as change?
or
B. How much money altogether did the two purchases cost?

106. When you're doing word problems, it's sometimes useful to "think aloud," to put into words your thought processes about what to do. It's also useful to hear or read other people's thought processes about word problems they've solved. Here's an example:

Rob earned $10 on Monday, and $4 more than that on Tuesday. How much did he earn altogether in the two days?

Here are the thought processes:

This is a "some and some more" problem. I want to add how much he made on the two days, to get the total that he made. But wait, I'm not told how much he made on Tuesday. How much he made on Tuesday is an "intermediate question." I've got to figure that out first. On Tuesday he made $4 more than Monday, and Monday he made $10. So on Tuesday, he made 10+4 or 14 dollars! Now that I've figured that out, I add the $10 he made on Monday to the $14 he made on Tuesday. 10+14=24. He made $24 altogether!

The intermediate question in this problem was

A. How much did Rob earn on Tuesday?
or
B. How much did Rob earn total, on both days?

107. Here's another example of a two-step word problem. Emily is 12 years old. Her sister Jillian is 5 years older than Emily was two years ago. How old is Jillian?

Here's what it might sound like if someone thinks aloud about this problem. "I'm asked how old Jillian is. I know she's 5 years older than Emily was 2 years ago. So if I only knew how old Emily was 2 years ago, I could solve the problem. So that's the intermediate question! I can figure that out: if she's 12 now, 2 years ago she was 12-2 or 10. Now since Jillian is 5 years older than that, Jillian is 10+5 or 15! And that's the answer to the problem!"

The intermediate question in this problem was:

A. How old is Emily now?
or
B. How old was Emily two years ago?

108. Here's another two-step word problem. Bill went to bet on the horse races. On Tuesday he lost $300. On Wednesday he lost $500. If he had $900 to start with, how much did he have after going to the races?

One person thinks about this problem as follows. "He started out with $900. If only I knew how much he lost altogether, I could figure out how much he has left. So that's the intermediate question! He lost $300 one day, and $500 the next, so he lost $300+$500 or $800 altogether. Now I can subtract to get the answer to the problem: he started out with $900, and lost $800, so he ends up with $900-$800, or $100!"

A second person thinks about this problem differently. "If only I knew how much he had on Wednesday morning before he went to the racetrack, I could get the answer. So let's let that be the intermediate question! He started with $900 and he lost $300 on Tuesday. So that means that Wednesday morning he had $900-$300 or $600 left! Then on Wednesday he lost $500 more, so after Wednesday's gambling he had $600-$500 left, or $100! And that's the answer to the problem!"

This section illustrates that

A. Sometimes there can be more than one intermediate question, and more than one reasoning path, that will take you to the correct answer in a logical way,
or
B. The first person asked the correct intermediate question and the second asked an incorrect one, even though the second person happened to get the answer right?

Chapter 6: Multiplication

Multiplication is repeated addition

109. Suppose that you have 3 packages of peanuts, and there are 5 peanuts in each package. How many peanuts are there in all?

One way to find out is to add the numbers up. There are 5 in the first package. The second package gives us 5 more; that makes 10. The third gives us five more, making 15. So there are 20 peanuts in all.

Another way of writing what we did to solve this is like this:

5+5+5=15

We wrote the number 5, 3 times, and put plus signs in between.

Because problems like this come up so often, where we have to add up the same number a certain number of times, we give a special name to this: multiplication. Instead of saying "We'll add 5, 3 times," we say "5 times 3," and write 5 x 3, or 5*3.

The point of this section is that

A. you get the same answer for a multiplication problem no matter what order the numbers are in,
or
B. multiplication means adding the same number a certain number of times?

110. When we make a picture of multiplication in our minds, it's often useful to arrange things in rows and columns. Here's a picture of 5 x 3:

u u u u u
u u u u u
u u u u u

We have 5 u's in each row, and we have 3 rows of them. Or we can also think, we have 3 u's in each column, and we have 5 columns.

The purpose of this section is to

A. prove that multiplication is repeated addition,
or
B. tell a way of making a picture of multiplication by making rows and columns?

Words for the numbers in a multiplication problem

111. We use certain words for the numbers in a multiplication problem, just as we do for other operations. If 6 x 4= 24, then 24, the answer, is called the product; 6 and 4 are factors. Or, we can say that 6 is the multiplicand and 4 is the multiplier. Because 6 x 4 is the same as 4 x 6, it doesn't make much difference which is the multiplicand and which is the multiplier; the word factors is therefore used more often. Any numbers multiplied together are called factors.

A summary of this section is that if 7 x 3 = 21, then

A. 7 and 3 are factors and 21 is the product,
or
B. there may be 7 sets of 3, or 3 sets of 7?

Skip counting

112. Suppose that we start with zero and add 5, then add 5 to what we get, then add 5 to what we get then. We get a series of numbers that looks like this:

5 10 15 20 25 30 35 40 45 50

This is called skip-counting by 5's. You are counting up, but skipping four numbers each time.

Suppose that instead of adding 5 each time we add 10. Then we get this series of numbers:

10 20 30 40 50 60 70 80 90 100

This is skip-counting by 10's. We skip 9 numbers each time we count.

When we skip-count by 2's, we skip 1 number each time:

2 4 6 8 10 12 14 16 18 20

The main purpose of this section was

A. to tell what skip-counting is, and give some examples of it,
or
B. to prove that when we skip count by adding a certain number each time, we skip one fewer than that number of numbers each time?

113. Since skip-counting is repeated addition, skip-counting and multiplication are almost the same thing, and we can solve lots of multiplication problems by skip-counting. Here's an example. We have 4 packages of peanuts, and each package has 10 peanuts. How many peanuts are there in all?

We can think: 1 package has 10, 2 have 20, 3 have 30, and 4 have 40. The answer is 40. Or if we want, we can skip count by 10s and put up one finger for each count. When we get to four fingers up, we're done. We count 10, 20, 30, 40. We have added up 10, 4 times. In other words, we have figured out 10 x 4. To multiply a number by another, you skip count by one of the numbers, for a number of counts equal to the other number.

What is 5 x 3? We can skip-count by 5's, for 3 counts. 5, 10, 15. The answer is 15. Or, we can skip-count by 3's, for 5 counts. 3, 6, 9, 12, 15. The answer is the same.

The main point of this section is that

A. it is OK to use your fingers to keep track of how many counts you've made when skip-counting,
or
B. when you multiply, you can skip count by one of the factors, for a number of counts equal to the other factor?

114. Let's think just for a second about the difference between rows and columns. A row goes from side to side, or horizontally, and a column goes up and down, or vertically. We speak of things going in rows and columns as an array, or a matrix. Here is a matrix with u's in it:

u u u u u
u u u u u
u u u u u

The matrix above has

A. 3 rows and 5 columns,
or
B. 5 rows and 3 columns?

Patterns you can find in the multiplication table

115. Let's do some skip-counting in an organized way. Let's skip-count by ones, for ten counts, and put the answers in the top row, then by twos, and put the answers in the second row, and then by threes, fours, and so forth, up to where we're skip-counting by tens. The result we get is the following matrix, commonly known as the multiplication table.

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

Because mathematics has to do so much with finding patterns, I want to spend some time examining the interesting patterns in the multiplication table. There are lots of them! Here’s one of them: we set out to skip count across the rows. But we wound up with the same pattern of skip-counting going down the columns! Please notice, if you haven’t already, that the first column goes exactly the same as the first row, the second column exactly as the second row, and so forth.

The pattern we would read when going across the fourth row is

A. the same as the one going down the fourth column,
or
B. different from the one going down the fourth column?

116. Here’s another interesting pattern you can see in the multiplication table. Let’s draw an imaginary diagonal line from the upper left corner of the table to the lower right corner. (The word “diagonal” means “from one corner to

another.") The numbers that go along this imaginary line are 1, 4, 9, 16, 25, 36, 49, 64, 81, and 100. Notice the differences between the numbers: from 1 to 4 is a jump of 3; from 4 to 9 is a jump of 5; from 9 to 16 is a jump of 7; and so forth. The pattern of differences goes 3, 5, 7, 9, 11, 13, 15, and so forth. The pattern is that the differences increase by two each time you move along the diagonal!

As if that were not enough, we can draw other sloping lines parallel to that main diagonal. (Parallel lines go beside each other in the same direction.) Look at the one just above the main diagonal: it goes 2, 6, 12, 20, 30, and so forth. And how about the one just below the main diagonal: it's the same pattern! Look at the pattern of differences along these sloping lines: the numbers jump up by 4, 6, 8, 10, 12, and so forth. The differences go up by two each time, just as they did for the main diagonal. How about for the sloping lines above and below the ones we just checked? The next one starts out 3, 8, 15, 24, 35. Again, the differences increase by two each time! If you check this out, you'll see that the same pattern holds for each of the upper-left to lower-right sloping lines!

The point of this section is that

A. The patterns for sloping upper-left-to-lower-right lines for the multiplication table are not predictable,
or
B. the increases in the numbers in the lines in the multiplication table that slope from upper left to lower right all follow the same pattern?

117. What about the diagonals going from lower left to upper right? If you want, figure out the pattern for these, before I tell you.

For these diagonals, the pattern is very much the same, only the differences, rather than becoming two more each time, become two less each time! Let's check it with the main lower-left-to-upper-right diagonal: the numbers are 10, 18, 24, 28, 30, 30, 28, 24, 18, 10. When we figure out the jumps between the numbers, we get 8, 6, 4, 2, 0, -2, -4, -6, -8. (When we got down to adding 0, the only way we could get a jump of two less was to subtract two, and then subtract 4, and so forth. It will be easier to talk about this later on when we allow ourselves to speak of "negative numbers.")

The pattern of numbers in the multiplication tables that goes from lower left to upper right is

A. not any regular pattern,
or
B. the jumps become two less each time?

118. Here's another interesting pattern you can find in the multiplication table; it's a pattern having to do with addition! Look at the top row, and find any two numbers that add together to give a third number in the top row. For example, 2 + 3 =5. Now look at the numbers just below those, in any given row, to see if they also add up. Under the 2 is 4; under 3 is 6; and under 5 is 10; 4+6=10! As we keep going down, we find 6+9=15, 8+12=20, and so forth. In every row, we get two numbers that add up to the third! The reason this works out this way is a property of numbers we'll discuss later called the distributive law.

The purpose of this section was to

A. help you to remember your multiplication facts,
or
B to point out two more interesting patterns you can find in the multiplication table?

119. The most commonly noticed pattern of all in this multiplication table is that you can use this table to find the product of numbers. If you want to know 6 x 3, you can look in the 6th row of the table (the one with the number 6 at the left), and then move in that row to the 3rd column (the column with 3 at the top). The place where the 6th row and the 3rd column cross each other tells you what 6 x 3 is.

You can find the same answer by looking in the 3rd row of the table and seeing where the 6th column crosses it.

If we have the same distance between columns and rows, then we get another pattern. We have a diagonal line running from upper left to lower right, formed by the "perfect squares" of numbers multiplied by themselves. Then we find other numbers duplicating themselves at the same distance on either side of this perfect square line! For example, when you find the number 54 on the upper side of this line, you find another 54 in the same position on the lower side of it.

One of the points made in this section is that

A. the imaginary line running from upper left to lower right is called a line of symmetry,
or
B. if you look where the 6th row crosses the 3rd column, you find what 6 x 3 is.

Why perfect squares are square

120. We call the numbers running along the diagonal from upper left to lower right perfect squares: 1 x 1 or 1, 2 x 2 or 4, 3 x 3 or 9, 4 x 4 or 16, and so forth. What is square about these numbers? Here's one answer.

Let's represent 2 x 2 by some u's, making 2 rows and 2 columns.

u u
u u

Let's represent 3 x 3 by 3 rows and 3 columns.

u u u
u u u
u u u

And let's represent 4 x 4 by 4 rows and 4 columns.

u u u u
u u u u
u u u u
u u u u

These arrays look pretty square, don't they? If they were little dots instead of u's, and if the distance between rows were exactly the same as the distance between columns, they'd be almost exactly square.

By contrast, let's represent something other than a number multiplied by itself: 5 x 2 for example. This looks like a rectangle, but not a square:

u u u u u
u u u u u

A square has an equal length on all four sides. So the representations of products of a number times the same number come out to be squares.

The purpose of this section is to

A. demonstrate that 5 x 2 = 10,
or
B. to explain why numbers multiplied by themselves are called “perfect squares?”

The commutative law of multiplication

121. The commutative law of addition says that, for example, 4+3 is the same as 3+4. 2+6 is the same as 6+2. It is the “order doesn’t make any difference” rule for addition.

We’ve already mentioned several times that the same thing is true for multiplication. Let’s try it for 5 x 10 and 10 x 5. If we skip count by 10’s 5 times, we say 10, 20, 30, 40, 50. We land on 50. If we skip count by 5’s 10 times, we say 5, 10, 15, 20, 25, 30, 35, 40, 45, 50. We land on 50 again. So we got the same thing for 5 x 10 as we got for 10 x 5. And we will always get the same thing for any pair of numbers multiplied together. The commutative law of multiplication is true for any two numbers.

Here’s a way to understand why the commutative law of multiplication is true. Let’s think about 5 x 3. We can represent this by 3 rows with 5 u’s in each row.

u u u u u
u u u u u
u u u u u

The three rows of 5 represent 3 fives, or 3 x 5.

But now let’s look at the same picture, only looking at the up-and-down columns rather than the side-to-side rows. We have 3 in each column, and we have 5 columns. The 5 columns represent 5 threes, or 5 x 3.

So we can look at the same picture and see it as 3 fives or 5 threes, depending upon whether we think about the rows or the columns.

This is the same for any multiplication fact.

u u u
u u u

This picture shows us that 2 x 3 is the same as 3 x 2.

The general rule is that order doesn't make a difference when you are multiplying two numbers.

The author puts u's in rows and columns in order to

A. show that you can use these to calculate any multiplication fact,
or
B. give you pictures showing that order doesn't make a difference when you multiply?

What we mean by "powers of 10"

122. We are going to start talking about multiplying numbers by 10, 100, 1000, or other "numbers like that." But we need a better name than "numbers like that," so we call them "powers of 10." Ten thousand, a hundred thousand, a million, ten million, and so forth are all powers of ten.

A "power" of a number means that number multiplied by itself a certain number of times. And all the powers of ten can be obtained by multiplying 10 by itself. For example, 100 is ten times ten. 1000 is 10 x 10 x 10, or ten to the third power. A million is 10 x 10 x 10 x 10 x 10 x 10, or ten to the sixth power.

Every number that is written by the numeral one, followed by a certain number of zeroes, is a power of ten.

Which of the following statements is a consequence of what this section told us?

A. The number 100,000 is a power of ten,

or
B. When we speak of ten to the sixth power, six is called an exponent?

Multiplying whole numbers by powers of 10

123. See if you find a general pattern in these products: 2 x 10 = 20; 3 x 10 = 30; 4 x 10 = 40, and so forth. Also, 11 x 10 = 110, 12 x 10 = 120, 13 x 10 = 130. 421 x 10 = 4,210. How do you multiply a whole number by 10? It works every time: the answer is the whole number with a zero added to the end of it!

What about multiplying by 100? 2 x 100 is 200. 3 x 100 is 300. 21 x 100 is 2100. Again, it works every time: you can multiply any whole number by 100 by attaching 2 zeros onto the end of it.

When we think about multiplying by 1000, we now have a pattern of patterns! To multiply by 10, add 1 zero; to multiply by 100, add 2 zeroes; to multiply by 1000, we add 3 zeroes, and so forth. The pattern keeps working: to multiply by a million (or 1,000,000) we add six zeroes.

The main idea of this section is that

A. multiplying by 10 or 100 or 1000 is something you'll want to do frequently,
or
B. to multiply by 10 or 100 or 1000, you attach one, two, or three zeroes to the end of the number?

The words multiple and factor

124. When you skip count by 3's, you land on certain numbers and you don't land on others. 3, 6, 9, 12, 15, 18, and so forth are the numbers you get when you multiply whole numbers by 3. These are called the *multiples* of 3. 5, 10, 15, 20, 25, 30, and so forth are multiples of 5.

When you skip count by 3's, you get the numbers for which 3 is a *factor*. 3 is a factor of 3, 6, 9, 12, 15, 18, and so forth. 5 is a factor of 5, 10, 15, 20, 25, and so forth.

Is 6 a factor of 24? Yes, because when we skip count by 6's, we land on 24: 6, 12, 18, 24. Is 6 a factor of 25? No, because we don't land on 25.

Is 28 a multiple of 7? Yes, because we can multiply 7 by a whole number and get 28. (The whole number is 4.) 4 and 7 are factors of 28; 28 is a multiple of 4; 28 is also a multiple of 7.

A consequence of the information in this section is that

A. if 10 x 6 = 60, then 60 is a multiple of 10 and 6, and 10 and 6 are factors of 60,
or
B. when you skip count, you generally start with 0?

Prime numbers and composite numbers

125. Any whole number can be expressed as itself multiplied by 1. For example, 6 x 1=6. So 6 and 1 are factors of 6.

But most whole numbers have factors other than themselves and 1. For example, 3 x 2=6, so 3 and 2 are also factors of 6. If a number has factors other than itself and 1, we call it a composite number. 6 is a composite number.

What about the number 7? There are no two whole numbers that multiply together to give 7, other than 7 and 1. So 7 is not a composite number. Any whole number that isn't composite is called prime, or a prime number. 7 is a prime number.

4 is a composite number, because 2 x 2=4. 12 is a composite number, because 3 x 4= 12 and 6 x 2=12. 5 is a prime number, because the only two whole numbers that multiply together to give 5 are 5 and 1.

We define prime numbers so that 0 and 1 are not considered prime numbers.

The first few prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, and 53. These are all numbers that are not the answer to any multiplication problem other than a times 1 problem.

Why do we care whether numbers are prime or not? The answer to that will come later!

The definition given in this section is that

A. a prime number is one with only two factors: itself and one,
or
B. the idea of prime and composite numbers applies only to whole numbers?

The associative law of multiplication

126. The associative law of addition said that, for example, 2 + (3+4) is the same as (2+3) + 4. This means that we get the same thing whether we add the first two numbers first and then add the second, or add the second two numbers first and then add the third. This is the "grouping doesn't make a difference" rule for addition.

Is the same thing true for multiplication? Does, for example, (2 x 3) x 5 give us the same thing as 2 x (3 x 5)?

If we try it out, we find that (2 x 3) x 5 is 6 x 5, or 30. 2 x (3 x 5) is 2 x 15, which is also 30. It works. (2 x 3) x 5 equals 2 x (3 x 5). Grouping doesn't seem to make a difference.

Here's a way of picturing this. I hope it will make clear why the "grouping doesn't make a difference" rule works for multiplication, every time.

This represents 2 x 3:

```
uuu
uuu
```

Now let's put 5 of those, right beside each other.

```
uuu   uuu   uuu  uuu  uuu
uuu   uuu   uuu  uuu  uuu
```

This represents 5 sets of that 2 x 3 number, or (2 x 3) x 5.

But look at the top row. Do you see that there are 5 sets of 3 in that top row? That is 3 x 5. There are two rows exactly alike. So there are two times 3 x 5, or 2 x (3 x 5).

So if we look at each group of 6, we see (2x3) x 5; if we look at the top row and then the bottom row, we see 2 x (3 x 5). We have the same number of u's in each case.

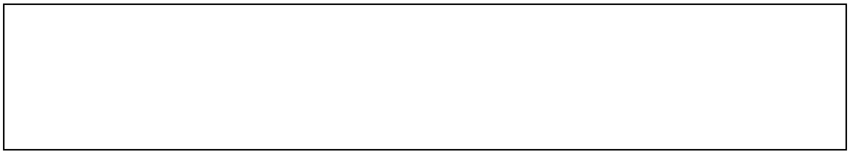
It's true all the time: when we multiply three numbers together, we can group together any two of them we want, and still get the same answer.

This section stated and illustrated the fact that

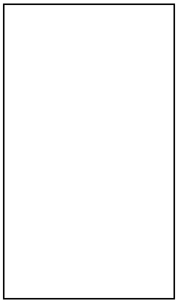
A. order doesn't make a difference when you multiply; this fact is called the commutative law,
or
B. grouping doesn't make a difference when you multiply; this fact is called the associative law?

Multiplication to find areas of rectangles

127. Let's talk a second about what rectangles and squares are. A rectangle can be shaped like this:

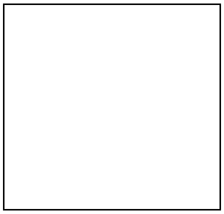


or maybe like this:



For rectangles, two things are always true. The opposite sides are the same length. Each corner of the rectangle forms a "right angle." (What's a right angle? It's the way the lines come together in the corners of the two rectangles above! It's also called a 90 degree angle. When two lines come together at a right angle, we say that the lines are *perpendicular* to each other.)

Here's a special type of rectangle, which is a square:



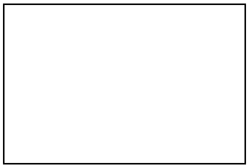
A square is a rectangle with all four sides equal. Every square is also a rectangle.

This section points out that

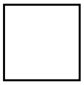
A. all squares are also rectangles,
or
B. all squares are also parallelograms?

128. Let's think carefully about what it means to ask what the area of a rectangle is.

Here's a rectangle.



And here's a little square, approximately one centimeter on a side. It's a "square centimeter."

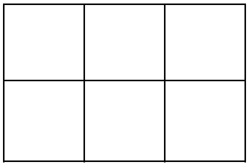


To ask, "What's the area of that rectangle" means that you want to know how many of the little squares it takes to exactly cover the whole rectangle. The answer tells how many "square centimeters" you'd need.

The main point of this section is that

A. the length of a rectangle is measured in centimeters and not square centimeters,
or
B. the area of a rectangle means how many square centimeters it would take to cover up the whole rectangle?

129. The question of how many square centimeters it takes to cover our rectangle is lots easier to answer when the rectangle is divided up into squares exactly the size of the little square.



In this case we can easily count the squares, to find that the area of the rectangle is six square centimeters. But we could also use multiplication. The square centimeters are arranged in rows and columns, in our drawing above. We have 2 rows, and 3 columns. So we have 2 x 3 or 6 square centimeters in all.

One of the main points of this section is that

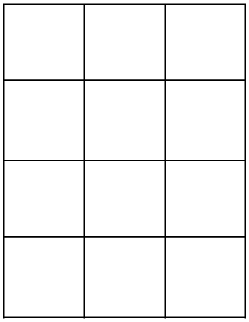
A. to get the area, you can multiply the number of rows by the number of columns of square centimeters in the rectangle,
or
B. you can find the area of shapes like circles, where the squares don't fit well?

130. In our picture just above, the number of columns is just the length of the rectangle, which is 3 centimeters. (How do we know the length is 3 centimeters? Because each edge of the square centimeter is one centimeter, and 3 of those edges fit perfectly along the longer side of the rectangle.) And the number of rows is just the width of the rectangle, which is 2 centimeters. So we could get the area by multiplying the length by the width. We would say, 2 centimeters x 3 centimeters = 6 square centimeters.

This section makes the point that

A. the area of a rectangle is equal to the length times the width.
or
B. the area of any figure is the number of "unit squares" it would take to exactly cover it.

131. Now let's look at another rectangle:



This one is 4 centimeters long and 3 centimeters wide. It also has 4 rows and 3 columns of squares. So the total number of little squares is 3 x 4, or 12.

The author presents this section as another example of how

A. area = length x width.
or

B. rectangles have right angles for their corners?

132. Let’s think for a second about the words “length” and “width.” In our 2 x 3 rectangle above, the length was the horizontal, or side-to-side distance, whereas in the second example, or 4 x 3 rectangle, the length was the vertical, or up-and-down distance. The distance of 3 centimeters was called the length in the first case, and the width in the second. How can this make sense? It’s just that the word “length” means the distance for the longer of the two sides of the rectangle, and the word “width” means the distance for the shorter of the two sides.

When deciding what to call the length and what to call the width of a rectangle,

A. rotating the rectangle around wouldn’t change which is called which,
or
B. rotating the rectangle around would change which is called which?

Introduction to the distributive law

133. Let’s represent the number 5 like this, as the sum of 2 + 3:

uu uuu

(2+3) is just another name for 5.

Now suppose we multiply this number by 4. We make four rows of what we had:

uu uuu
uu uuu
uu uuu
uu uuu

You’ll notice that there are 4 2’s and 4 3’s in the answer. (Look at the four rows of twos on the left, and the four rows of threes on the right.) So when we multiply 4 by (2 +3), the answer is 4 2’s plus 4 3’s, or 4 x 2 + 4 x 3. In quadrupling (2+3), we quadrupled 2 and we quadrupled 3, and added the results together. The fact that we quadrupled both the 2 and the 3, not just one of them, is a consequence of the distributive law.

Here's another example. (1 + 2) is another name for 3. Here's a picture of it.

u uu

If we multiply it by 3, we get this:

u uu
u uu
u uu

And we notice that there are 3 ones and 3 twos. (Look at the three rows of one on the left and the three rows of two on the right.) So

3 x (1+2) = 3 x 1 + 3 x 2

This is another example of what?

A. the associative law,
or
B. the distributive law?

134. What does the distributive law really tell us? It says that when we are tripling the quantity (1+2), we can't just triple the one, or triple the two; we have to triple both of them. Or in pictures, if we want three times as much as this:

u uu

we can't just make this:

u uu
u
u

and we also can’t just make this:

u uu
   uu
   uu

but we have to make this:

u uu
u uu
u uu.

The distributive law is a very important law for the subject of algebra. We’ll come back to it later on, and think more about what it means and how it’s used. But for now, it’s good just to realize that if we want to make several copies of a number that’s in two parts, we have to duplicate both parts, not just one of them.

The author suggests that

A. if you know that you have to copy a whole row rather than just part of it, you know all there is to know about the distributive law,
or
B. there is much more to do before fully understanding the distributive law?

Multiplication problems giving the number of sets, and the number in each set

135. Many multiplication problems are like this one:

If you have 4 sets, and there are 3 in each set, how many do you have altogether?

To get the answer, you multiply the number of sets by the number in each set, to get the total number.

u u u
u u u
u u u
u u u

The above picture of u’s shows 4 sets of u’s with 3 in each set. There are 4 x 3 or 12 altogether.

But the language is often just a little different, so you have to figure out that there are a certain number of sets and a certain number in each set. The problem doesn’t exactly tell you. Here’s an example.

Someone buys 4 books; each one costs 3 dollars. How much do the books cost in all?

To imagine our 4 sets of 3, we could imagine 3 one dollar bills laid out in a row, one for each book. There would be four rows of three dollar bills, just as there are 4 rows of three u’s above.

The main point of this section is that

A. to get the price of 4 books at $3 per book, you multiply 4 x 3.
or
B. whenever you are given the number of sets and the number in each set, and you need to find the total, you multiply the number of sets by the number in each set?

136. It’s good practice, when you are given multiplication word problems, to identify which of the numbers your given is the number of sets and which is the number in each set. Let’s look at some examples.

Mary travels 4 miles an hour. She goes for 5 hours. How many miles does she go in all?

In this problem each hour represents a set of 4 miles gone, and there are 5 of those sets of 4 miles. So 4 is the number in each set, and 5 is the number of sets.

Here's another problem:

Each computer weighs 4 kilograms. There are three computers. What does the scale read if you put all three computers on it?

In this problem each computer represents a set of 4 kilograms that will register on the scale. There are 3 sets of those 4 kilograms (because there are 3 computers). So 4 is the number in each set, and 3 is the number of sets.

Here's another problem:

Marian got 100 points on each of 8 tests. How many points did she score altogether?

In this problem each test represents a set of 100 points. There are 8 such sets. So 100 is the number in each set, and 8 is the number of sets.

Finally, here's one more, for you to figure out:

John can type 20 words in one minute. How many words can he type in 5 minutes?

A. 5 is the number in each set, and 20 is the number of sets,
or
B. 20 is the number in each set, and 5 is the number of sets?

137. For some problems, it's possible to think in two different ways about which of the numbers is the number of sets, and which is the number in each set. For example:

Teresa arranges chairs in rows and columns. There are 7 columns and 10 rows. How many chairs are there in all?

We can think of each column as a set. If we do, there are 7 sets, and 10 in each set. But we can also think of each row as a set. If we do, there are 10 sets, with 7 in each set. Fortunately we get the same answer either way.

This section gave an example of a problem where

A. the ideas of number of sets and number in each set just don't apply,
or
B. you can think in two different ways about what is the number of sets and the number in each set?

138. In all the examples we have looked at recently, the multiplication problem has given us a certain number of sets, and a certain number in each set. To find the total number, we multiply:

total number = number of sets x number in each set.

You will find this pattern occurring over and over in word problems involving multiplication.

The purpose of this section was

A. to summarize the last few sections,
or
B. to introduce a new idea about what the total number equals?

The algorithm for multiplication of whole numbers

139. What is an algorithm? It's a set of steps that you go through, in much the same way each time, that lets you solve a certain type of problem.

Let's think about the algorithm for multiplying a two-digit number by a one digit number. How about 15 x 3. We'll start by writing the problem vertically:

$$\begin{array}{l} 15 \\ \underline{x3} \end{array}$$

Now we multiply 3 ones x 5 ones and get 15 ones. Just as with addition, when we come out with ten or more ones in the ones place, we put down the number of ones and "carry" the tens.

1
15
$\underline{x3}$
5

Now we multiply 3 by 1 ten, to get 3 tens, and we add the 1 more ten that we carried, to get a total of 4 tens. We write that in the tens' place of our answer, and we're done!

1
15
$\underline{x3}$
45

In the algorithm for multiplying a two digit number by a one digit number, you

A. start by multiplying ones by ones,
or
B. start by multiplying ones by tens?

140. Now let's think about how you multiply 15 by 20.

15
x $\underline{50}$

The first thing we do is just to bring down the 0 as a placeholder. This has the effect of multiplying the answer we get by 10. This means that if we just multiply 15 by 5, with the added zero in effect we've multiplied it by 50. Why? Because 10 x 5 is 50.

Here goes 0 down as a placeholder:

15
x $\underline{50}$
0

Now we just multiply 15 by 5: first 5 x 5, gives 25; put down 5 and carry 2. 5 x 1 = 5, plus 2 is 7.

$$\begin{array}{r} 15 \\ \text{x}\ \ \underline{50} \\ 750 \end{array}$$

Multiplying 15 x 50 was

A. Very much like multiplying 15 x 5, only with sticking a zero on the end of the answer,
or
B. Very different from multiplying 15 x 5?

141. Now we've multiplied 15 x 3 and 15 x 50. What about multiplying 15 x 53? The thing that makes the algorithm work is the fact that to multiply 15 x 53, you can first multiply 15 x 3, then multiply 15 x 50, and then add those products. This is a consequence of what we call the distributive law.

When we multiplied 15 x 3 we got 45. When we multiplied 15 x 50 we got 750. So when we multiply 15 x 53, we're going to get 45 + 750, or 795. Let's show how we organize it when we multiply:

We start out by multiplying 3 x 15, just as we did before:

$$\begin{array}{r} 15 \\ \text{x}\,\underline{53} \\ 45 \end{array}$$

Then we make another row, where we multiply 50 by 15, just as we did before, writing a 0 in the ones' place and then multiplying 15 by 5:

$$\begin{array}{r} 15 \\ \text{x}\ \ \underline{53} \\ 45 \\ \underline{750} \end{array}$$

Now we just add 45 and 750, and we're done!

$$\begin{array}{r} 15 \\ \text{x}\ \ \underline{53} \\ 45 \\ \underline{750} \\ 795 \end{array}$$

When we multiplied 15 x 53, we

A. first multiplied 53 by 10 and then by 5 and then added the results,
or
B. first multiplied 15 by 3 and then by 50, and then added the results?

142. When you do multiplication of two two-digit numbers, here is the order in which you multiply the four digits: lower right times upper right, lower right times upper left. (Then bring down a zero.) Then lower left times upper right, then lower left times upper left. (Then add the two products.)

When you multiply two two-digit numbers, you start with

A. lower left times upper left,
or
B. lower right times upper right?

143. What happens when you multiply, say, two three digit numbers? You continue the same pattern.

$$\begin{array}{r} 123 \\ \text{x}\ \ \underline{456} \end{array}$$

To do this one, first you would multiply 6 by 123.

$$\begin{array}{rr} & 123 \\ \text{x} & \underline{456} \\ & 738 \end{array}$$

In the next step, you would multiply 50 (the 50 in four hundred fifty-six) by 123. You’d be sure to put a zero in the ones’ place so that when you multiply by 5, you’re really multiplying by 50.

$$\begin{array}{rr} & 123 \\ \text{x} & \underline{456} \\ & 738 \\ & 6150 \end{array}$$

In the next step, you would multiply 400 (the 400 in 456) by 123. You’d be sure to put 2 zeroes in the ones’ and tens’ places, so that when you multiply by 4, you’re really multiplying by 400.

$$\begin{array}{rr} & 123 \\ \text{x} & \underline{456} \\ & 738 \\ & 6150 \\ & 49200 \end{array}$$

Finally, you add up the three numbers you’ve gotten (which are called the “partial products) to get the final answer!

$$\begin{array}{rr} & 123 \\ \text{x} & \underline{456} \\ & 738 \\ & 6150 \\ & \underline{49200} \\ & 55088 \end{array}$$

Chapter 7: Telling Time With Analog Clocks and Watches

144. Keeping up with time is a pretty complicated operation, when you think about it. There are 60 seconds in a minute, 60 minutes in an hour, 24 hours in a day, around 30 days in a month, 12 months in a year, 7 days in a week, and about 365 days in a year. This is a lot different from, for example, the metric system of measuring lengths, where 1000 millimeters or 100 centimeters are in a meter, 1000 meters in a kilometer, and so forth; all the numbers we use to change units are multiples of 10. Why are things so inconsistent when we measure time?

This section

A. wonders why there are so many different numbers involved in changing one unit of time to another,
or
B. explains how to tell time with an analogue clock?

145. There are some good reasons for some of these numbers. The earth happens to revolve around the sun about every 365 days, and this makes the cycle of the seasons start over that frequently; that's why we start over on that schedule when counting years rather than something like every 100 days. The 12 months come from the fact that the moon goes through its cycle (from full to dark to new and back to full again) about 12 times in a year. This comes from the fact that the moon revolves around the earth about 12 times a year.

As to why the day is divided into 24 hours (2 sets of 12 that repeat themselves), with 60 minutes in an hour and 60 seconds in a minute, people aren't so sure. Some people think that ancient people counted, not just with the fingers, but with the joints of their fingers. Suppose you use your thumb to count the three joints of your 4 remaining fingers. You get 12. If you put up one finger on the other hand for each 12, you put up 5 fingers after you've counted to 5*12, or 60.

This section presented

A. the reason why clocks run clockwise,
or

B. the reasons for some of the numbers that we use to convert units of time to other units of time?

146. Other people point out that 60 is a great number if you want to divide it into equal parts, because it can be divided evenly into any of the following numbers of parts: 2, 3, 4, 5, 6, 10, 12, 15, 20, and 30!

Here are another couple of theories. A minute is about the length of time that most people can for example run at top speed without being slowed down by being too out of breath. When we are resting, our hearts beat somewhere close to 60 times in that interval, if we're in good shape (which our ancient ancestors probably were, not having tv's, video games, and machines to do their work for them).

This section offers

A. a couple more theories about how it came about that 60 seconds are in a minute, and why a minute is the length of time it is,
or
B. the fact that mechanical clocks were invented somewhere around the 14th century in Europe?

147. More and more clocks and watches make it very easy to tell the time: you simply read off the numbers the timepiece displays. 12:45 is twelve forty-five, 10:22 is ten twenty-two, and so forth. Such time-tellers are called digital clocks or watches, because they show the time using digits.

I wonder how much longer anyone will use, or learn to use, analog clocks and watches? These are clocks that look like this:

So what time is it, by this clock? We look at the shorter hand to tell the hour. The hand is between the 10 and the 11, so it's 10 something – it hasn't reached 11 yet. The longer hand is on the 3, so we say 10:15. Why didn't we say 10:03? Because we have to remember that for minutes, each of the large numbers printed on the clock represents 5 minutes, not one minute. So if we multiply the number the big hand is pointing to by 5, as in 3 * 5, we get the number of minutes to read out. This is the reason why knowing the five times tables really well helps in telling time with an analog clock.

What does 10:15 mean? It means that 10 hours and 15 minutes have gone by since either midnight (in which case it's 10:15 a.m.) or since noon (in which case it's 10:15 p.m).

Which is a summary of this section?

A. If the shorter hand is between two numbers, the hour is the smaller number; if the longer hand is pointing to a number, the minutes are equal to 5 times that number.
or
B. a.m. and p.m. stand for *ante meridiem* and *post meridiem*, which mean *before midday* and *after midday* in Latin.

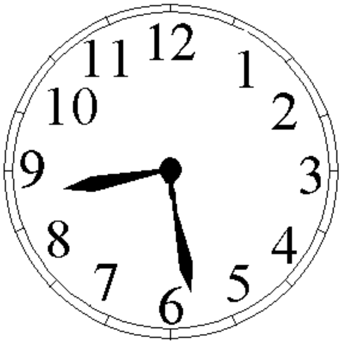
148. Sometimes we need to look at the minute hand to be sure what hour we are on with the hour hand.

A quick glance at the clock above might lead us to think that the hour hand is pointing to the 2. But the minute hand is at 50 minutes, which means it's almost to a certain hour, but not quite. Thus the hour hand must be almost to the 2, but not quite. The time is 1:50.

The challenge illustrated in this section was

A. reading the second hand,
or
B. figuring out what hour to read when the hour hand is almost to a certain number but hasn't reached it?

149. With analog clocks, we have another challenge when the minute hand is between numbers. What time is the next clock pointing to?



It’s somewhere between 8:25 and 8:30. If I make 4 imaginary marks for 8:26, 8:27, 8:28, and 8:29, trying to divide the space between the 5 and 6 evenly, I come up with the answer that it’s 8:28. Do you get the same thing?

This section illustrates

A. the mental process you go through when the minute hand is between numbers and you don’t have the minutes marked off on the clock,
or
B. the mental process you go through to decide that with the hour hand between the 8 and 9, it’s 8 something until you get all the way up to 9?

150. Let’s look at another clock.

Here we don’t have numbers on the clock, but that’s not the main problem. When we’ve practiced enough, we know that the hour hand is a little past the 2 and the minute hand is almost to the 11.

But wait. There’s a problem with this clock. When the minute hand is getting close to 12, but not quite, the hour hand should be almost to a certain number, not a little past it. The clock-maker made the hour hand go a little bit farther than it should. Even so, we can still figure out the time – it’s about 1:54.

This section brought out the point that

A. if you’re making an analog clock, you have to make sure that the hour hand and the minute hand are consistent with each other,
or

B. some clocks have no marks at all, and you just have to put all the numbers on in your imagination?

151. Before we leave the subject of telling time, here's something strange about the most common U.S. system of telling time that you probably haven't thought about.

Why do we call midnight or noon 12:00 rather than 0:00? We're wanting to start all over at midnight or noon. At 12:30 in the afternoon, how many hours and minutes have gone by since noon? Not 12 hours 30 minutes, but 0 hours 30 minutes.

Let's say you time people who are running a race. Some people run the race in 53 minutes, and some take an hour and 6 minutes. If we write this in hours and minutes, we write the faster people's times as 0:53, and the slower people's times as 1:06. In order to get to 1 hour, you have to cover all the time between 0 and 1. That's the way that really would make sense in expressing the time with clocks, too.

We could say that a time expresses the number of hours and minutes that have gone by since midnight if we're in the a.m., and the number of hours and minutes since noon if we're in the p.m. But this isn't true for a time like 12:30 pm. 12:30 pm means that 12 hours and 30 minutes have gone by since midnight, not since noon. Then we get to 1 pm and all of a sudden this means that 1 hour has gone by since noon.

I told you earlier a rule that you multiply by five the number that the minute hand is pointing to. But what about when the minute hand is on the 12? 5 * 12 equals 60, but the number of minutes when the minute hand points to 12, for example at 10:00, is not 60, but zero. We're acting like there's a 0 up at the top of the clock, even though it isn't there! So starting at 12 rather than 0 doesn't make a lot of sense, if you ever stop to really think about it.

In this section the author

A. makes a case for dividing a minute into a hundred seconds rather than 60,
or
B. makes a case for starting with 0:00 at midnight rather than 12:00?

152. Most people in the U.S. also don't know that the most common system of keeping time in most of the rest of the world does not have this problem. In what

is called the 24 hour clock, which is the international standard of time notation, at midnight you start with 0:00. After noon, which is 12:00, you keep going to 13:00 and 14:00 without starting over until the following midnight. Finally you get to 23:59 (which is 23 hours, 59 minutes after the previous midnight) and in one minute, you start over at 00:00. We might call 00:30 “zero zero thirty.”

The other main advantage of the 24 hour clock, of course, is that you don’t have to remember to write “a.m.” or “p.m.” to specify whether something happened in the morning or afternoon. This makes it preferred by medical people and military people, among others.

The author in this section reveals that

A. the problem he raised in the previous section has already been solved, in the system of time keeping most commonly used in the world,
or
B. computer programmers find the 24 hour clock less cumbersome to use than the 12 hour system?

Chapter 8: Division

The meaning of division

153. Suppose that we have 15 peanuts total. We are going to put them into packages with 5 peanuts in each package. How many packages will we need?

This problem asks, how many 5's are in 15. In other words, how many 5's do we have to add together to get 15?

Let's let each u stand for a peanut. Let's draw 15 of them:

u u u u u u u u u u u u u u u

Now let's draw a line under sets of five of them:

$\underline{u\ u\ u\ u\ u}\ \ \underline{u\ u\ u\ u\ u}\ \ \underline{u\ u\ u\ u\ u}$

We see that there are 3 sets of 5. So the answer to our question is 3 packages.

The question, "How many threes are there in 15" is the same as saying "What is 15 divided by 3." Division means asking how many of a certain number are in another number. So we write,

15 / 3 = 5

We can also write division like this:

$15 \div 3 = 5$

or like this: $\begin{array}{r} 5 \\ 3\overline{)15} \end{array}$

The main idea of this question is that

A. you can do division problems by making u's and drawing lines under them, or

B. division answers the question, “How sets of a certain number are in another number?”

154. But “how many sets of one number are in a second one” is not the only question that division answers. Let’s look at this question:

If you divide 15 cards equally among 5 players, how many will each player get?

Now we’re given the total number, and the number of sets (5), and we’re asked for the number in each set!

Figuring out the distinction between these two types of problems will give good exercise to the understanding of division. But the same division operation will answer both of them.

Here’s another where the unknown that we are asked to find is the “number in each set” and not the “number of sets”:

If you have 15 peanuts and put them equally into 5 packages, how many will be in each package?

In this problem about the 15 peanuts divided equally into 5 packages,

A. 15 is the total, 5 packages is the number in each set, and we’re asked for the number of sets;
or
B. 15 is the total, 5 packages is the number of sets, and we’re asked for the number in each set?

155. We’re discovering that there was one type of multiplication problem, but there are two types of division problems. For multiplication, every problem said, “We have this number of sets, and this number in each set. How many are there altogether, (or how many total)?”

In these multiplication problems, we’re looking for the total. In division problems, however, we’re told the total. We can be

told the total, and told the number in each set, and asked for the number of sets,

or we can be

told the total, and told the number of sets, and asked for the number in each set.

The main point of this section was that

A. there is one type of multiplication problem but two types of division problems,
or
B. the total divided by the number of sets equals the number in each set?

Multiplication and division fact families

156. We spoke earlier about fact families for addition and subtraction. Multiplication and division are relatives, just as addition and subtraction are. Let's look at the following batch of u's:

u u u
u u u

We can think in several ways about what this picture tells us. If we look at the 3 columns with 2 in each column, we find that 3 x 2 =6. If we look at the 2 rows with 3 in each row, we find that 2 x 3 =6. If we take all 6 things and divide them into 2 rows, we have three in each row, or 6/2=3. And if we take all 6 things and divide them into 3 columns, we have 2 in each column, or 6/3=2. So we have four facts in our family. It is the same for any other multiplication or division fact: there are three "relatives" that come along with each fact.

What is a consequence of the message of this section?

A. The three relatives of 3+5=8 are 5+3=8, 8-5=3, and 8-3=5
or
B. The three relatives of 6 x 4= 24 are 4 x 6=24, 24/6=4, and 24/4=6?

Divisions that don't come out evenly

157. Let's consider the following problem:

There are seven people. We want to divide them into teams with three people on each team. How many teams can we make?

Let’s draw one u for each person:

u u u u u u u

And now let’s draw lines under sets of 3 u’s, to represent dividing the people into teams of 3:

$\underline{\text{u u u}}$ $\underline{\text{u u u}}$ u

We see that we have two sets of three people. So, in a sense, the answer to our question is “We can make two teams.” But if we want to communicate that there is one person left over, we can say that the answer to our division problem is 2, remainder 1. This answer gives us a clue that if two more people show up, we’ll have enough for another team; we don’t need to wait for three more to show up. We write the answer to our problem as

2 r 1

where r stands for remainder.

This section dealt with the situation where

A. we are dividing things into groups of a certain number, and there are some left over;
or
B. we are doing a division problem where the total number is less than the number we are dividing by?

158. Division problems with remainders often give us some thinking to do about whether to “round down” or “round up.” In the previous problem, when our division problem came out to be 2 remainder 1, the answer was that we could form 2 complete teams. We “rounded down” 2 remainder 1 to just 2.

But now consider this problem.

We have some cars that will carry 3 passengers. How many cars will it take to carry 7 passengers to where we need to go?

When we divide, we come out to same answer that we got before: 2 remainder 1. We fill 2 cars with three people, and then we have 1 person left over. But in this situation, we can't just leave the other person waiting; we need to provide transportation for that person too. So we need one more car, even though it's only partially filled. The answer is that we need 3 cars. In this problem we rounded 2 r 1 up to 3 rather than down to 2.

We decided that a result of 2 r 1 gave us an answer of 3 in this problem, whereas in the previous problem, a result of 2 r 1 gave us an answer of 2.

What point is this section trying to get across in dealing with division problems with remainders?

A. that sometimes you end up splitting the remainder too, so that you have a fractional answer,
or
B. we need to think about the situation the problem asks about, and visualize what is going on, in order to decide what answer fits the problem when there is a remainder?

Words for the numbers in division problems

159. Let's say you're dividing 17 by 5, and you get 3 r 2. Let's assign words to all these players in the division problem. 17 is the dividend; 5 is the divisor, 3 is the quotient, and 2 is the remainder.

With division problems involving whole numbers, the dividend is the biggest number of all. This is the "total" that you're parceling out into pieces. The divisor is what you're dividing by. The answer you get in a division problem is the quotient. What's left over is the remainder.

The point of this section was to

A. prove that the quotient times the divisor, plus the remainder, equals the dividend,
or

B. define the words dividend, divisor, quotient, and remainder?

The number you can't divide by

160. Suppose that someone asks us, "What is 5 divided by 0?" Let's ask this as a missing factor problem. 0, when multiplied by some missing factor, is supposed to equal 5. But there's a problem with finding this missing factor: any number multiplied by zero is zero! How are we going to come up with a factor which, when multiplied by 0, gives 5? We can't do it. The problem has no solution. We run into the same problem when we divide 4, or 7, or 8, or 1/2, or any other number by zero. So we say that division by zero is impossible, and that the answer to such a problem has no meaning, or is undefined.

The main point of this section is that

A. division problems can be thought of as "missing factor" problems, as in "What factor, multiplied by the divisor, gives the dividend?"
or
B. division by zero is meaningless or undefined?

161. What about the following problem: 0 divided by 0? Let's ask this, also, as a missing factor problem. 0, when multiplied by some missing factor, is supposed to equal 0. Let's guess and check, or try certain numbers and see what we get. How about 1? 1 x 0 is 0. Maybe we've found the answer. But how about 2? 2 x 0 is 0 also, so 2 looks like a correct answer also. How about 3, 5, 7, 28, 49, 17/25? All of those numbers, when multiplied by 0, also give zero. In fact, any number you can name, when multiplied by 0, gives 0, and is an answer to the missing factor problem.

So with 0 divided by 0, we have the opposite problem as when we tried to divide 5 by 0. When we tried to divide 5 by 0, no number could work as a missing factor. When we tried to divide 0 by 0, we got that all numbers could work as missing factors. Either way, we don't get an answer that is meaningful or helpful. So we say that division by zero "is not allowed."

The main idea is

A. division by zero is not allowed, even when the dividend is zero,

or
B. 5 divided by zero does not have a solution because no number times zero gives five?

162. Let's think about division in the situation where we are giving out food to people. Suppose we have 4 apples, and we want to divide them equally among 4 people. We can do this with no problem by giving 1 apple to each person, and we can represent our thinking by saying 4/4=1. Or if we want to give out the 4 apples equally and all we have is 1 person, again we have no problem. We just give all four to that one person, and we represent our thinking as 4/1=4. But what if we are supposed to give out 4 apples equally to 0 people? Now we have 4 apples to get rid of, but no one to give them to. We have an impossible task. We represent this by saying 4/0 is undefined or impossible.

The purpose of this section was to

A. give a concrete example of a situation where division by zero doesn't make sense,
or
B. to make very clear the difference between dividend and divisor?

How about 0 divided by something?

163. To continue speaking about dividing up apples: suppose we have 0 apples, and we are suppose to give each of 4 people an equal share. Now there's no problem. Each person gets 0 apples. We have divided up 0 very nicely into 4 equal parts. We represent what we've done by saying 0/4=0.

Let's think about division as a missing factor problem. If the problem is 0/4, then 4 is one factor, and we're looking for a missing factor to multiply by 4 to give 0. There's only one answer, and that's 0. So we've got one and only one right answer, and we run into no difficulties at all.

So even though we can't divide BY zero, we can easily divide zero by any other number.

Another way of stating the point of this section is that

A. because there is no factor which, when multiplied by zero, gives 10, 10 divided by zero is undefined,
or
B. although zero doesn't work as the divisor (the number we divide by), zero works fine as a dividend (the number to be divided).

The division algorithm

164. The division algorithm is the series of steps you use in dividing one number by another. The steps are: divide, multiply, subtract, and bring down. This series of steps gets repeated until the problem is finished. Here's an example of what these steps mean:

$$
\begin{array}{r}
18 \\
2\overline{)36} \\
\underline{2} \\
16 \\
\underline{16} \\
0
\end{array}
$$

When dividing 2 into 36, we start by asking a "divide" question: how many 2's are in 3? There's only 1, so we write 1 above the 3. Next, we multiply 1 by the divisor 2, and write the result under the 3. Next we subtract: 3-2 is 1. Then we bring down the next digit, 6, to make 16. Now we start the steps over, and see how far we can get with another cycle of divide, multiply, subtract, and bring down.

We divide 2 into 16; the answer to this is 8. We write 8 above the 6. Next we multiply 8 by 2 to get 16. We write that 16 under the 16 that was already there. We subtract again, 16-16, to get 0. And now, since there's nothing more to bring down, we're done. So we've used the "divide-multiply-subtract-bring down" algorithm, which is called long division, to find that 36 divided by 2 is 18.

This section

A. gave an example of the division algorithm,
or

B. explained why the division algorithm works?

The division algorithm when the divisor has two digits

165. When the divisor has two digits, it's usually necessary to do some guessing by estimating. Let's give an example.

$$\begin{array}{r} 36 \\ 24\overline{)864} \\ \underline{72} \\ 144 \\ \underline{144} \\ 0 \end{array}$$

We start out by asking, "How many 24's are in 86?" Now, most people haven't memorized the multiplication tables for 24! So there's sometimes some trial and error involved in figuring out the first number to put down. But for now, let's imagine that we guess correctly on the first time, and figure that there are about 3 24's in 86. We write our 3 above the 6, and then multiply 3 x 24. We write the result, 72, underneath the 86. Then we subtract. We get 14, and then we bring down the 4. We've done one cycle of divide, multiply, subtract, and bring down.

Now we divide 24 into 144. Let's imagine that we guess correctly again on the first time, that there are 6 24's in 144. We write the 6 above the 4, then multiply 6 x 24 to get 144. We subtract to get 0, and there's nothing more to bring down, so we're done. The quotient is 36.

In the example given in this section,

A. the steps in long division still were divide, multiply, subtract, bring down,
or
B. we made several wrong guesses before getting the right one?

166. When using the division algorithm, how do we know that the number we try for the quotient is a correct one? The test comes when we multiply that number by the divisor. If the product we get is greater than the number we're dividing into, so that we can't subtract, we know our guess is too large. We know that we should try a smaller number. On the other hand, let's say we multiply, and then

subtract. If we get a difference that's greater than the divisor, we know our guess was too small. The divisor would "go into" the dividend at least another time.

Let's look at an example of this guess-and-check process with a simple division problem, one we probably wouldn't overshoot or undershoot on.

$$\begin{array}{r} 3\_\, \\ 4\overline{)96} \\ \underline{12}\, \end{array}$$

We start by asking, how many 4's are in 9? Let's suppose we guess 3. We multiply 3 x 4, and write 12 under 9. But 12 is greater than nine, so we know we've overshot. We should pick a smaller number.

This section gives an example of

A. What happens when the number we pick to put in the quotient is too big,
or
B. what happens when you multiply incorrectly?

167. Now let's look at what happens when the number we pick to put in the quotient is too small.

$$\begin{array}{r} 1\_\, \\ 4\overline{)96} \\ \underline{4}\,\, \\ 5\,\, \end{array}$$

We ask, "How many 4's are in 9," and we guess 1. But when we multiply 1 x 4, getting 4, and subtract 9-4, we get 5. We have 5 left over. But since 5 is greater than the divisor, 4, we know that there's at least one more 4 in 9 than we thought there was! So we'd better try a bigger number.

This section gave an example of

A. What happens when the number we pick for the quotient is too small,
or

B. what happens when the number we pick for the quotient is too big?

168. So now we pick a bigger number to try.

$$\begin{array}{r} 2\underline{} \\ 4\overline{\smash{)}96} \\ \underline{8} \\ 1 \end{array}$$

When we tried 2, and multiplied by 4, we got 8, and when we subtracted, we got 1. 1 is not over 4, so we know we didn't guess too small. And 8 is not greater than 9, so we know we didn't guess too big. Our number for the quotient was just right. To repeat, we keep guessing and checking, if necessary, until we can subtract in the subtraction step, but the difference we get in the subtraction step is less than the divisor.

This section

A. explained clearly why the difference in the subtraction step can't be bigger than the divisor,
or
B. gave an example of what happens when you pick the right number to put in the quotient: you can subtract, but you get a number smaller than the divisor?

Why do we go from left to right in dividing?

169. Figuring out why the division algorithm works the way it does is a pretty tough task. Let's think about it some. For adding, subtracting, and multiplying, you start at the right end of the number, or the ones' place. But when we divide, we start at the left end of the dividend.

For example, in

$$\begin{array}{r} 23 \\ +\underline{19} \end{array}$$

The first thing we do is add 3 ones and 9 ones. Why? So that if we need to carry over a ten to the tens' column, we can do that before we've already added the tens.

But with

$2\overline{)36}$

we start at the left end of the 36, by dividing the 2 into 3. Why?

This section

A. explained why you start at the left end when dividing,
or
B. posed the question of why you start at the left end when dividing, without answering it yet?

170. In trying to understand the division algorithm, let's imagine we are dividing money evenly between two people. We have 3 ten dollar bills (represented by t's) and 6 one dollar bills (represented by u's, that stand for "units.")

Here's what we start out with:

t t t   u u u u u u
Now let's start by giving one ten to one person, and one to another. We'll represent that by writing a t above and below the part that hasn't been divided up yet.

first person: t

not divided yet: t u u u u u u

second person: t

So how are we to divide that ten that is left? Let's get change, and turn it into ten ones.

first person: t

not divided yet: u u u u u u u u u u u u u u u u

second person: t

Now we have 16 ones that need to be divided. We divide those by dealing out 8 to each person, like this:

first person: t u u u u u u u u

not divided yet:

second person: t u u u u u u u u

And we're done. Each person has gotten a ten and 8 ones, or $18.

This section

A. used a division bracket to represent dividing money,
or
B. used t's and u's to stand for ten dollar bills and one dollar bills, to represent dividing money between two people?

171. Please hold in memory what we did with the t's and u's in the last section. Now look at what we do with the division bracket:

$$\begin{array}{r} 18 \\ 2\overline{)36} \\ \underline{2} \\ 16 \\ \underline{16} \\ 0 \end{array}$$

We start with the three tens, and we give 1 to each person. When we subtract 3-2 we find that we have 1 ten left over, still to be divided. When we bring down the 6 ones, we are thinking that we have 16 ones left to be divided. When we write 8 in the quotient, we are dividing those 16 into 2 equal parts. When we subtract 16 from 16 we are checking to make sure that there are none left over. Do you see that our division algorithm is just doing the same thing we were doing when we divided up the money?

The purpose of this section is to

A. help you remember the steps in the division algorithm,
or
B. show you that the steps in the division algorithm are the same ones that we used in dividing money?

172. Now please go back to the question, why do we start at the left when we divide? The answer, when we are dividing tens and ones, is that if the number of tens can't get divided evenly, we'll want to lump any tens that get left over with the ones. Suppose we divided the ones first. Then suppose we divided tens, and had some left over. We'd want to add those extra tens to the ones, but we would have already divided up some of the ones! This sort of reasoning explains why in division, we start dividing the bigger places first, or in other words, why we go from left to right.

This section attempts to

A. wrap up the explanation of why we go from left to right when dividing,
or
B. demonstrate that you can check division by multiplying?

Using the division algorithm when there are remainders

173. What happens when our final subtraction in a division problem yields a difference that is not zero? For example:

$$\begin{array}{r} 4 \\ 3\overline{)14} \\ \underline{12} \\ 2 \end{array}$$

We have two options at this point. We can say that the answer is 4, with 2 left over undivided, or 4 r 2. Or, we can divide the 2 into 3 equal parts, which, as we will see when we read about fractions, becomes 2/3; so our answer becomes 4 2/3.

A summary of this section is that

A. when a division comes out uneven, you either take the last difference and make it the remainder, or you express it as a fraction by dividing it by the divisor,
or
B. a whole number plus a fraction, like 4 2/3, is called a mixed number?

Checking division with multiplication

174. If 10 / 5 = 2, then 2 * 5 = 10. The quotient times the divisor should give the dividend. We can use this fact to check a division problem. If the quotient times the divisor doesn't equal the dividend, something is wrong.

What about when there is a remainder? We multiply the quotient by the divisor and add the remainder; the result should be the dividend. For example:

11/5 = 2 r 1   and to check, 2 * 5 =10; 10+1=11.

If the answer comes out to be a mixed number like 5 1/2, then our simple rule holds: the quotient times the divisor should equal the dividend. In our example, 5 1/2 times 2 equals 11.

This section said that

A. the quotient times the divisor, plus the remainder if there is one, should equal the dividend, and you can use this fact to check division,
or
B. division is the inverse of multiplication?

Using 0 as a place-holder when dividing

175. Here’s one more tricky thing about division. Suppose we “bring down” and we’re ready to divide again, but the divisor is bigger than the number we’re dividing into? That situation clues us to put a 0 as a place-holder in the quotient. Let’s look at an example.

$$\begin{array}{r} 1\underline{} \\ 9\overline{)918} \\ \underline{9} \\ 0\;1 \end{array}$$

We’re dividing 9 into nine hundred eighteen. 9 into 9 goes 1 (that’s the divide step); 1 times 9 is 9 (that’s the multiply step); 9 minus 9 is 0 (that’s the subtract step) and we bring down 1 (that’s the bring down step). Now we’re ready to divide again. How many 9’s are in 1? The answer is zero, with one left over! So we write a 0 in the quotient space, and just continue as per usual. 0 times 9 is 0 (the multiplication step); 1 minus 0 is 1 (the subtraction step) and bring down the 8 (the bring down step). Now we divide 9 into 18 and get 2. Doing this results in a 0 used as a place-holder in the quotient.

$$\begin{array}{r} 102 \\ 9\overline{)918} \\ \underline{9} \\ 01 \\ \underline{0} \\ 18 \\ \underline{18} \\ 0 \end{array}$$

The point of this section was that

A. division is the inverse of multiplication,
or
B. when you’re doing the division algorithm, and your divisor is bigger than the number you’re trying to divide into, you put 0 in the quotient as a place holder?

Chapter 9: Order of operations

Multiplication and division come before addition and subtraction.

176. Suppose you have math expressions like this:

3 x 4 + 5

or 2+8/2? .

What do you do first?

People have agreed on what these expressions mean. They have agreed that you do multiplication or division before addition or subtraction. For example, they have agreed that in expressions like

3 x 4+5

you do the multiplication first. So 3 x 4 = 12. When you add 5 to it, you get 17.

A point made by this section was that

A. when you have an expression like 4+3 x 8 you do the multiplication first, not the addition,
or
B. when you have two or more multiplications or divisions next to each other, you go from left to right?

Parentheses go first

177. But what if you WANT someone to add 4+5 first, and then multiply what they get by 3? People have also agreed upon a way of telling each other, “We want you to do the addition first.” They tell this by putting parentheses around the

addition problem. The parentheses mean, “Don’t worry about the usual rule of doing multiplication before addition. This is an even more powerful rule, that says do what’s in parentheses before anything else.” So

3 x (4+5)

means that you first add 4+5 to get 9. Then you multiply 3 x 9 to get 27. So

3 x (4+5) is NOT the same as 3 x 4 + 5!

So far we’ve talked about the rules that

A. you go from left to right,
or
B. you do what’s in parentheses first, then multiplication, then addition?

178. How would you figure out the following?

7 – 2 + 3?

Suppose we decided to start with 2+3. We’d get 5. Then we subtract 7-5 and get 2. We’d get the wrong answer! Why? Because people have agreed that when you have a string of additions and subtractions or multiplications and divisions, you go from left to right. The correct answer would be gotten by thinking: 7-2=5, 5+3=8. 8 is the correct answer, not 2.

The point of this section is

A. when you are doing additions and subtractions, or multiplications and divisions, you go in order from left to right,
or
B. if you want someone to do something other than go from left to right, there’s a way to tell them to do that?

179. Please look again at 7 – 2 +3. Suppose you really wanted someone to add the 2 and 3 together first, and then subtract what they got from 7? Again, there’s a way to tell them to do it, and the way is the same: parentheses. If you see

7 –(2+3)

you should do the addition first, and then take 7-5 to get 2.

The point of this section is that

A. the orders of operations are the way they are just because people have agreed to do them that way, not because of some fact about nature,
or
B. if you want someone not to use the “left to right” rule, you can use parentheses to tell them to do what’s in parentheses first.

180. Here’s another example of how the “left to right” rule applies. What is

30 / 3 x 5?

If we go left to right, we divide 30 by 3 to get 10, and then multiply that 10 by 5, to get 50.

But if we decide to do 3 x 5 first, we get 15, and when we divide 30 by 15, we get 2. The answers are quite different! The first answer is correct.

Again, if we want to communicate to people that 3 x 5 is to go first, we use parentheses. If we ask people to do

30 /(3 x 5)

The correct answer is 2.

The point of this section is that

A. the left to right rule is important for problems with multiplication and division as well as with addition and subtraction,
or
B. the left to right rule is not really needed when all you have is a series of multiplications, or a series of additions?

181. Suppose you have the following expression:

3+2+7+6

If you try it out, you'll see that it doesn't matter whether you go left to right, right to left, or in any order you want. Why is the "left to right" rule unimportant with this series of numbers to be added? Because of the commutative and associative laws of addition, otherwise known as the "order doesn't make a difference" and "grouping doesn't make a difference" rule. Because order and grouping don't make a difference, you can do it however you want!

In the same way, if you are asked to find

3 x 2 x 4 x 5

you get the same answer no matter what order you go in, because of the commutative and associative law of multiplication.

But our "left to right" rules became important when there was subtraction or division in the expression. Why? Because subtraction and division are NOT commutative and associative. 5-3 is NOT the same as 3-5, and 6/3 is NOT the same as 3/6. 5-(3-2) is NOT the same as (5-3)-2. 12/(4/2) is NOT the same as (12/4)/2.

The main idea of this is that

A. we have to be careful about the left to right rule when there are subtractions or divisions in the problem, because these operations are not commutative,

or

B. parentheses come first, no matter what?

Be careful about "Please excuse my dear Aunt Sally"

182. "Please excuse my dear aunt Sally" is a way that lots of people remember the order of operations in math. If you take the first letter of each word in this sentence, you get the letters P, E, M, D, A, and S. The P stands for parentheses; the E stands for Exponents (which we'll talk about later); the M and D stand for multiplication and division, and the A and S stand for addition and subtraction. So this helps us remember that we do what's in parentheses first, then multiplications or divisions, and then additions or subtractions.

What’s the main point of this section?

A. That exponents mean repeated multiplication, just as multiplication means repeated addition,
or
B. That there’s a way people have used to remember what order to do the operations when we have expressions with several operations?

183. However, you have to be careful about “Please excuse my dear aunt Sally.” What it DOESN’T help you remember is that you don’t necessarily do addition before subtraction, or multiplication before division. When there is a choice between doing a multiplication or division first, you go from left to right. And when there is a choice between doing an addition or subtraction first, you also go from left to right.

So if you are given

10 – 7 +3,

you don’t do the 7+3 first, even though A comes before S in Please Excuse My Dear Aunt Sally. You do the 10-7 first, because when there are additions and subtractions, you go from left to right. So 10 – 7 + 3 comes out to 6, not 0.

This section gives you the information to decide that 12 –5 + 4 is equal to

A. 3
or
B. 11?

184. Likewise, when you have a choice between a multiplication and a division, you go from left to right. You don’t necessarily do the multiplication first because m comes before d in the sentence. So if you are given

12 /3 x 4

you go from left to right and get 4 x 4 or 16, not 12/12 or 1.

So: multiplication and division are on equal footing in the contest to see which is done first, just as addition and subtraction are. You choose between multiplication and division by the "left to right" rule, just as you choose between addition and subtraction by the "left to right" rule. (Multiplications and divisions always come before additions and subtractions.)

The point this section and the previous one are making is that

A. when choosing between addition and subtraction, or between multiplication and division, for which one to do first, you go from left to right,
or
B. when you are in doubt, it doesn't hurt to put in parentheses to make it crystal clear which operation is to be done first, as for example by writing (12/3) x 4, even though you really don't need the parentheses?

185. So here is the true order of operations:

1. Parentheses
2. Exponents
3 and 4. Multiplication and division (tied). When choosing between these, go from left to right.
5 and 6. Addition and subtraction (tied). When choosing between these, go from left to right.

Since multiplication and division are tied, and addition and subtraction are tied, it makes just as much sense to say, "Please excuse, dear, my silly aunt!"

The purpose of this section is to

A. display the information you've already been told, in a different format,
or
B. introduce new information about the order of operations?

Chapter 10: Even and Odd Numbers

186. Earlier we talked about factors and multiples. An even number is a multiple of 2. It is a number that has 2 as a factor. An odd number is not a multiple of 2, and does not have 2 as a factor.

So the numbers we land on when we skip count by 2 are even numbers. The numbers we don't land on when we skip count by 2 are odd numbers.

The even numbers start out like this:

0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 46...

The odd numbers start out like this:

1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41, 43, 45...

When you are naming odd or even numbers in order, you go up by jumps of 2 in each case. Which of the following is correct?

A. If you start with 0, you'll get even numbers, and if you start with 1, you'll get odd numbers,
or
B. If you start with 1, you'll get even numbers, and if you start with 0, you'll get odd numbers?

187. Here are some things to notice about odd and even numbers.

If you start counting (by ones) with the number 1, the numbers you land on take turns between being odd and even. For example, 1 is odd, 2 is even, 3 is odd, 4 is even, and so forth.

The even numbers all end in 0, 2, 4, 6, or 8. (By "end," I mean the even numbers have a 0, 2, 4, 6, or 8 in the one's place.) This is how you can tell quickly if a number is even.

The odd numbers all end in 1, 3, 5, 7, or 9. This is how you can tell quickly if a number is odd.

According to the rule we just talked about, the number 2,379,461 is an

A. even number,
or
B. odd number?

188. Here are some more facts about even and odd numbers:

Every even number is the answer to an addition fact that is a "double." For example, 1+1=2. 2+2=4. 3+3=6. 4+4=8. The answers we are getting start out 2, 4, 6, 8.

If we picture odd and even numbers, the even numbers are the ones we can divide into two equal groups.

If we picture odd and even numbers, the even numbers are the one where each one in the group has a partner, and none are left over.

Let's look again at u's. When there are 4 in all, each can be paired up with a partner:

uu
uu

This is the same way it is for all even numbers. But if we have 5 of them, when we pair them up, one is left without a partner:

uu
uu u

This is the way it is for all odd numbers.

According to what this section told us, if we had 2,639 u's, and we paired them up with partners as much as we can,

A. every one would have a partner,
or
B. one would be left without a partner?

Adding even and odd numbers

189. When you add 2 even numbers, what kind of number do you get? Try it for a few pairs of numbers, like 6+4, 8+6, 10+8.

When you add 2 odd numbers, what kind of number do you get? Try it for a few pairs of odd numbers, like 3+5, 3+7, 5+9.

When you add an odd number to an even number, what kind of number do you get? Try it for a few pairs of numbers where one is odd and one is even, like 6+3, 1+8, or 5+6.

What rules do you come up with? See how they compare with these:

even+even=even

odd+odd=even

even+odd=odd.

We can summarize the rules of this section by saying that when adding two numbers,

A. if both are odd, or both are even, the result is even; if one is odd and one is even, the result is odd,
or
B. the sum of two numbers is always even?

190. Let's make some pictures to understand why these rules always hold.

When we have two even numbers, all of them are paired up to begin with. So all the things can keep their partner, and all will still be paired up. Let's show this with 4+2.

Here are 4: uu uu

And here are 2: uu

Here are 4+2: uu uu uu

Do you see that everyone in the group stays paired up?

This section illustrated that

A. when you add two even numbers, each thing has a partner to begin with, and they all keep their original partners; therefore the sum is even.
or
B. when you add an even and an odd number, there's only one thing without a partner, and it can't get a partner; therefore the result is odd?

191. When we have two odd numbers, there is one in each group without a partner. Those two can pair up with each other when the two numbers get together.

Let's show it with 3+5.

Here's 3: uu u (One doesn't have a partner.)

Here's 5: uu uu u (One doesn't have a partner.)

Here's 3+5: uu u uu uu u But now the two unpartnered ones can pair up, so that everybody has a partner. Join those two singles in your mind, and you have the following:

uu uu uu uu

So you get an even number. It's like that when you add any two odd numbers.

This section illustrated that

A. when you add an even and an odd number, there's only one thing without a partner, and it can't get a partner; therefore the result is odd,
or
B. when you add two odd numbers, there are two things without partners, and they can join up with each other; therefore the result is even?

192. Now let's add an even and an odd number. Let's do it with 4 and 3.

Here's 4: uu uu (Everyone has a partner.)

Here's 3: uu u (One doesn't have a partner.)

When we put them together, we get this: uu uu uu u

The one without a partner stays without a partner, because everyone else was already paired up. So we get an odd number. It's the same way every time we add an even and an odd number.

This section illustrated that

A. odd plus odd equals even,
or
B. odd plus even equals odd?

Subtracting odd and even numbers

193. You could go through exactly the same sort of reasoning that we used for adding odd and even numbers, to figure out the rules for subtracting odd and even numbers. Or, you can just think of simple examples, such as
4-2=2
5-3=2
7-4=3
8-3=5

The rules for subtracting odd and even numbers correspond exactly to those for adding. Here are the rules:

even minus even=even
odd minus odd = even
odd minus even = odd
even minus odd = odd

A way of summarizing the rules of this section is that

A. when subtracting, if both numbers are odd, or both numbers are even, the answer is even; if there is one of each, the answer is odd,
or
B. an odd number plus an odd number is an even number?

Multiplying odd and even numbers

194. An even number is a multiple of 2. Therefore, any number multiplied by an even number is also a multiple of two. The only numbers which come out to an odd product are two odd numbers. So

even times even = even
even times odd = even
odd times even = even
odd times odd = odd

The purpose of this section was to

A. define multiples of two
or
B. state the rules for how numbers come out odd and even when multiplying?

What about division?

195. Here's an even number divided by an even number which gives an even number: 8/4=2. But here's an even number divided by an even number which gives an odd number: 6/2=3. So there is no rule for even divided by even.

An odd number divided by an even number doesn't come out evenly – it comes out with a remainder or a fraction. 13/4=3 r1 or 3 1/4.

An odd number divided by an odd number has to give an odd answer, as in 9/3=3. And an even number divided by an odd number has to give an even answer, as in 12/3=4.

This section

A. stated what rules there are about odd and even when dividing,
or

B. defined odd numbers as multiples of two, plus one?

How you can use these rules

196. What good are these rules? One practical use for them is that you can use them to check your addition and subtraction. Suppose you add 17 and 8, and you get 26. Could this be right? No, because 17 is odd and 8 is even, so the answer has to be odd. So I'd better figure out a better answer!

The point of this section is that

A. The difference of two odd numbers is always even,
or
B. you can use the rules about sums and differences of odd and even numbers to check your arithmetic?

Chapter 11: Literal Numbers, or Variables

Using letters to represent numbers

197. Lots of times we let things stand for other things. If a valentine says "I ♥ you," we figure that the ♥ stands for the word "love." The 50 stars on the U.S. flag stand for the 50 states. If we let the letter A stand for the area of a rectangle, and the letter L stand for the length of that rectangle, and the letter W stand for the width of that rectangle, we are letting letters stand for numbers. Letters that stand for numbers are called "literal numbers."

When we can decide among *various* different choices what numbers those letters will stand for, we also call such numbers "variables."

The main point of this section is that

A. The fifty stars on the U.S. flag stand for the 50 states in that country.
or
B. We can let letters stand for numbers.

198. Numbers that are always equal to one certain amount are called *constants*. Numbers like 4, 27, 5/3, 0.08, and so forth, are constants. There are some numbers that we call by letters that are also constants. One famous one is pi, which is named by a Greek letter that looks like this: π. It's equal to a little over 3.14. There's another constant that mathematicians call e; it's equal to a little over 2.718. So not all numbers that are named by letters are variables. But most of the time, we let a letter represent a number because we want to be able to substitute various different numbers for that letter.

The main point is that

A. Pi is equal to 3.14,
or
B. Most of the time, literal numbers are variables.

199. Some of the things we've already talked about are a lot easier to say if we allow letters to stand for numbers. Let's think about the commutative law of

addition. This law says, “When you are adding two numbers, you get the same result no matter what order you add them in.” If we wanted to list the specific facts that were true as a result of this law, we would have a job that goes on forever. We’d maybe start out by saying,

$1+2=2+1$
$1+3=3+1$
$1+4=4+1$

but we could go on forever before we ever got to

$2+3=3+2$
$2+4=4+2$
$2+5=5+2$

So clearly, listing all the specific facts that are true is not the way we want to state the commutative law of addition.

Why don’t we state the commutative law by using specific numbers, the ordinary type, like 2 and 6 and 81?

A. Because the law is true for some of those numbers and not others,
or
B. because we’d have to keep going forever to say all the facts that the commutative law tells us about specific numbers?

200. So in order to save ourselves from having to write forever, we use literal numbers or variables. Here’s what we say: Let’s let a represent any number at all, and b also represent any number at all. Then

$a+b = b+a$.

Now we’ve stated the commutative law, by writing very much shorter than forever! We can stick any number we want for a, and any number we want for b, and the statement is true. Let’s do it for $a=2$ and $b=9$. We have to make sure to substitute carefully, so that every time we see “a” in our expression, we stick in 2 in its place, and every time we see “b” we stick in 9 in its place. We get

2+9=9+2, which is indeed a true statement.

On the other hand, if we let a=36 and b=14, we get

36+14=14+36, another true statement.

We can get an unlimited, or infinite, number of true statements, just from

a+b=b+a.

This section made which point about literal numbers?

A. Using letters lets us represent unknown numbers, and then use the laws of algebra to find out what those numbers are,
or
B. By using letters, we can write things very quickly that would take us forever to write if we wanted to write every single example with ordinary numerals?

201. Let's represent some other laws by letters. How about the associative law of addition? This told us that grouping doesn't make a difference when we add. For example, when we add

2+(3+4) (and we do the 3+4 first, because of the parentheses)

we get exactly the same number as when we do

(2+3) +4 (doing the 2+3 first.)

We get 9 in each case, don't we?

If we want to use literal numbers to express this law, we say that for any three numbers a, b, and c,

a+(b+c)=(a+b)+c.

Now we've again represented every possible example of this law, without having to keep writing forever.

The information in this section that has to do with literal numbers is that

A. the distributive law says that when we multiply by a sum, we can multiply by each of the terms in that sum and add up the products that we get,
or
B. for any three numbers a, b, and c, $a +(b+c)=(a+b)+c$.

202. Let's introduce a little custom that mathematicians started a long time ago. In mathematics we very often multiply variables. We could write these as a x b, where x stands for multiplication. But this can get confusing, because x happens to be used very frequently as a variable. We could use the same sign that lots of computer programs use for multiplication, the asterix, and represent a times b by writing a*b. (This book will use the asterix as a multiplication sign, a lot, from here on out!) And we could use an older custom, of using a dot to stand for multiplication. But mathematicians have decided that if we just write a letter standing for a number, next to another number, without any sign, we're to assume that the numbers are to be multiplied together. So

5a

means, "multiply 5 by whatever number a stands for."

and

xy

means, "multiply the number that x stands for, by the number that y stands for."

Remember that we can't leave out the multiplication sign for ordinary numerals. If we tried to write 5 x 15 as 515, we'd get everybody mixed up.

The main point of this section is that

A. when you put a variable next to another number without any sign, you assume that you multiply the numbers,
or
B. It is difficult to make a raised dot for multiplication when you are typing?

203. Here are another couple of mathematical customs. When we write any number, literal or otherwise, next to a parenthesis, even with no multiplication sign, we mean for the number to be multiplied by what's in parentheses. So $a(b+c)$ means you multiply a by the sum of b and c; $5(2+3)$ means you multiply 5 by the sum of 2 and 3. $(10)(2)$ means ten times two.

One more custom: when we want to represent an ordinary numeral times a literal number, we put the ordinary numeral first. This custom is seldom written down anywhere; you just see expressions written that way. So if we want to say "the product of 5 and the number that x represents," we would usually write $5x$ and not $x5$.

What were the two points this section made?

A. you multiply if a number is written next to a parenthesis with no sign, and you put ordinary numerals first when you are multiplying ordinary numerals by literal numbers,
or
B. the distributive law has to do with addition, and it also has to do with multiplication?

204. Now that we have taken care of that detail, it will be easy to state the commutative and associative laws of multiplication using literal numbers. Here we go with the commutative law, or the "order doesn't make a difference law of multiplication":

$ab=ba$

Notice that we left out any multiplication sign?

Now here we go with the associative law of multiplication, the "grouping doesn't make a difference" law:

$a(bc)=(ab)c$

One of the points made in this section is that

A. if we state the distributive law using literal numbers, we say $a(b+c)=ab+ac$,
or
B. if we state the associative law of multiplication using literal numbers, we say $a(bc)=(ab)c$?

205. Let’s state the distributive law using literal numbers. Let’s first review by using a picture of 5(3+4) and show that this is 5x3 + 5x4:

Let’s make 5 copies of 3+4:

uuu uuuu
uuu uuuu
uuu uuuu
uuu uuuu
uuu uuuu

We see 5 3’s (in the columns on the left) and 5 4’s (in the columns on the right). So the picture tells us that 5(3+4) = 5 x 3 + 5 x 4.

To make this general, let’s talk about any three numbers a, b, and c rather than 5, 3, and 4:

$a(b+c)=ab+ac$

This is the general statement of the distributive law!

The main conclusion of this section is that

A. of the 5 laws we’ve talked about (commutive law of addition and multiplication, associative law of addition and multiplication, and the distributive law) the distributive law is the only one with two different signs in it,
or

B. using literal numbers, we state the distributive law by saying that $a(b+c)=ab+ac$?

206. Now let's use literal numbers (also known as variables) to tell how you find the perimeter of a triangle. Let's call the lengths of the three sides of the triangle a, b, and c. Let's call the perimeter of the triangle P. How do we use these literal numbers to communicate the fact that to find the perimeter, we just add up the lengths of the three sides? We just write

$P=a+b+c$.

This rule is called a formula. Formulas tell you how to find something if you know some other things. In this case, the formula tells you how to find the perimeter if you know the lengths of the 3 sides of the triangle. Notice that this statement only makes sense when we explain, in words, what P and a and b and c stand for. Whenever we use formulas, we should explain what the letters stand for.

One major point of this section is that

A. when we have a formula, we should tell what the letters stand for,
or
B. the perimeter of a pentagon, which has 5 sides, would be found by the formula $P=a+b+c+d+e$?

207. Let's use literal numbers to tell how to find the perimeter of a rectangle. As with any other figure, we just add up the lengths of the sides. So if P is the perimeter, and L is the length, and W is the width, we imagine ourselves walking around the rectangle, and we get

$P=L+W+L+W$

because there are two sides that are L units long, and two sides that are W units long. (Remember that the opposite sides of a rectangle are equal.)

We can make this formula a little shorter by first applying the "order doesn't make a difference rule" to get

$P= L+L+W+W$,

and then realizing that when we add a number to itself, that's the same as multiplying it by 2 (for example 6+6 is the same as 2 x 6). So our short version of the formula is

$P=2L+2W$

What's the main conclusion of this section?

A. the opposite sides of a rectangle are equal,
or
B. the formula for the perimeter of a rectangle is $P=2L + 2W$, where L is the length and W is the width?

208. Finally, let's represent the formula for the area of a rectangle using literal numbers. We want to communicate the fact that to get the area, you just multiply the length times the width. Here we go:

$A=LW$

where A=area of the rectangle, L=length of the rectangle, and W=width of the rectangle.

The main point of this section, expressed in literal numbers, is that

A. $A=LW$ is the formula for the area of a rectangle,
or
B. $A=WL$ could also be the formula for the area of a rectangle, because of the commutative law of multiplication?

209. So far we've used literal numbers (or variables) to make more general statements than we could make with ordinary numerals. We have used literal numbers because it would take too much work (usually an unending, or infinite, amount) to write out all the possible examples using numerals.

There's another reason for using variables. Sometimes we know something about a number, but we don't know what that number is. If we represent the number with a literal number, and write down what we know about it, we can figure out what it is.

Here's an example of this. I'm thinking of a number, but I'm not telling what it is. This number plus 5 equals 7. How can we write this with literal numbers?

Let's call our mystery number X. The problem says that the unknown number plus 5 equals 7. We can write that like this:

$X+5=7$.

This expression containing an equals sign is called an equation. It's an equation with an "unknown," namely X.

To solve the equation, we find a value for X that makes the equation true. If we try several numbers, or use the strategy of "guess and check," we find that only one number works. Let's try 0: $0+5$, or 5 doesn't equal 7; so X doesn't equal 0. Let's guess 1: $1+5$, or 6, doesn't equal 7; that doesn't work, but we're getting closer. Let's guess 2: $2+5=7$; it works; so X has to be equal to 2.

The method we used here to solve the equation was to

A. just try a few values for X until we found one that made the equation true,
or
B. subtract the number 5 from both sides of the equation?

Functions

210. Imagine a machine, where we put in a certain number, and we get a certain number out. For example, if we put in 5, we get out 6. If we put in 3, we get out 4. If we put in 7, we get out 8. What is the machine doing? It's adding one to each number that comes in, to get the one that comes out. It's a +1 machine. Such imaginary machines are called *functions*.

Here's another way of saying this +1 function, by using literal numbers. Let's call x the number that goes into the machine. Let's call y the number that comes out. The number that comes out, or y, is one more than the number that goes in, or x. If we write “y is one more than x” in mathematical symbols, we write

$y=x+1$

If x is 10, $y=10+1$ or 11.
If x is 7, $y=7+1$ or 8.

We can substitute whatever number we want for x, and get the value for y.

The main point of this section is that

A. we can use literal numbers to write functions, which are relations between an “input number” and an “output number,”
or
B. we could have used letters other than x and y if we had wanted to?

211. Now let’s imagine a different machine. We put in 2, and we get out 4. We put in 3, and we get out 6. We put in 5, and we get out 10. What is this function doing? It’s doubling the number that goes in, or multiplying it by 2, to get the number that comes out. It’s a “times 2” machine. If we use literal numbers, we may say that the y that comes out is 2 times the x that goes in. Or:

$y= 2x$

If we put in 4 for x, y comes out 8. If we put in 7 for x, y comes out 14.

The purpose of this section was to

A. teach you some methods of solving equations,
or
B. give you another example of a “functional relationship” between numbers?

Addition and subtraction can undo each other

212. Suppose we have a plus 2 function. We put in 1, we get out 3. We put in 2, we get out 4.

What kind of function machine would we have to have, to take the numbers we get out of the plus 2 function, and make them like they were at the start? In other words, what kind of function "undoes" the plus 2 function?

In other words, for the plus 2 function, we put in 1, we get out 3. Suppose we wanted to take that 3 and get our 1 back. What sort of function would we need?

It would be a minus 2 function, wouldn't it? 3 minus 2 gets us back to 1.

The minus 2 function "undoes" the plus two function.

Here's another way of thinking about why this is true. On the number line, the plus 2 function moves you two jumps to the right from wherever you start. The minus two function moves you two jumps back to the left. So after a plus two and a minus two, you end up where you started.

A function that "undoes" another function is an "inverse" function.

What's the inverse function for the plus 5 function? It's the minus 5 function.

What's the inverse function for the minus 3 function? It's the plus 3 function.

A main point this section makes is that

A. an inverse function is one that undoes whatever the other function did,
or
B. a "plus 5" function would NOT be an inverse of a "minus 2" function?

Multiplication and division can undo each other

213. Let's think about a "times 3 function." If we put in
1, we get out 3;
2, we get out 6;
3, we get out 9.

What function will let us take the numbers we got out, and get back to the numbers we put in?

We want to
put in 3, and get out 1,
put in 6, and get out 2,

put in 9, and get out 3.

The function that will let us do this is the “divided by 3” function. 3 divided by 3 is 1, 6 divided by 3 is 2, and 9 divided by 3 is 3.

So the “divided by 3” function undoes the “times 3” function. The two functions are inverse functions of each other.

This section gave an example of

A. a division function that was the inverse of a multiplication function,
or
B. a function where you first add and then multiply, which had an inverse function where you first divide and then subtract?

Chapter 12: Rounding and Estimating With Whole Numbers

214. Suppose that you divide 12,394 by 435, and you come out with 2,849. Does that answer “make sense?” One person might be able to look at the answer and quickly say, “No, that answer could not be right,” whereas a second would have to divide it out all the way. The first person has probably had more experience in rounding and estimating. Without skill in rounding and estimating, it’s hard to know whether the answers you get for problems “make sense” or not.

The point of this section was

A. to teach you the steps you go through in rounding,
or
B. to give a sales pitch for estimating and rounding as ways of telling whether your answers make sense?

215. How do you estimate answers? First you round the numbers in the problem. 12,394 divided by 435 is a fairly tedious problem. But 12,000 divided by 400 is much simpler. To estimate the answer to this easier problem, all we have to do is to recall one of our elementary division facts, 12/4=3, and then, to figure out how many zeroes our answer should have.

A summary of this section is that

A. to estimate an answer, we first round, then use an elementary math fact, and then figure out how many zeroes the number should have,
or
B. to use “front end” rounding you simply turn into zeroes the digits to the right of the ones you’re interested in?

216. Here’s the traditional method of rounding. First, you decide on the “place” to which you wish to round. This means, you decide whether you want to round to the nearest ten, hundred, thousand, ten thousand, or whatever. So if you want to round 761,289 to the nearest hundred thousand, you’d get 800,000, and if that’s

close enough for you, then that's what you want to round to. On the other hand, you could round the same number to the nearest thousand, and get 761,000.

If you wanted to round 88, you would round to the nearest 10, or 90. If you wanted to round 1,375 you might want to round to the nearest 100, and get 1,400. Or you could round 1,375 to the nearest thousand, and get 1000.

How do you decide whether to round to the nearest ten or hundred or thousand or what? It depends on how accurate you want your answer to be. Usually when estimating, you want to leave only one or two digits that aren't zeroes. A common reason for rounding is to see if your answer "makes sense," and you can usually do that by turning almost all the digits of your starting number into zeroes!

The point of this section was to

A. explain how you decide whether you increase by one, or leave the same, the "rounding digit."
or
B. describe the first step in rounding, which is to decide what place you want to round to?

217. "Rounding to the nearest hundred" means that the digit in the hundreds place is the first one you don't necessarily turn into a zero. This is called the "rounding digit." For example, if we are going to round 2,746 to the nearest hundred, the 7, which is in the hundreds' place, is the first digit we don't necessarily turn into a zero; it is called the rounding digit. If we're rounding 2,746 to the nearest thousand, the 2, which is the number in the thousands' place, is the rounding digit. All the digits to the right of the rounding digit get turned into zeroes. The rounding digit itself either stays the same, or moves up one.

The purpose of this section was to

A. tell the difference between front end rounding and traditional rounding,
or
B. define the phrase, "rounding digit"?

218. When you round, you leave the rounding digit as it is if the digit to the right of it is 4 or less. You move the rounding digit up one if the digit to the right of it

is 5 or more. For example, when rounding 285 to the nearest 10, the rounding digit is 8, because 8 is in the tens' place. Because the digit to the right of 8 is 5 or greater, we'd change the 8 to a 9, and round up to 290.

If you are asked to round 43,785 to the nearest hundred, 7 is the rounding digit, because 7 is in the hundreds' place. You'd round it up to 8 because the digit to the right of it is 5 or greater. So 43,785 would be rounded to 43,800.

If you are rounding 43,785 to the nearest thousand, 3 would be the rounding digit, and you would round to 44,000. If you round 43,785 to the nearest ten thousand, 4 would be the rounding digit, and you would round to 40,000.

The purpose of this section was to

A. explain how to figure out how many zeroes to put on your estimate,
or
B. to give a rule about whether to move the rounding digit up or leave it the same, and give examples of this rule?

219. We just stated that the rule was, to move the rounding digit up one when the digit to the right of it was 5 or more and leave it the same when the digit to the right was 4 or less. Why does this rule make sense? Because when we are rounding to the "nearest ten," for example, our rule tells us the multiple of ten that is nearest our original number. For example, if we are rounding 88, we are choosing between 80 and 90 as nearby multiples of 10 to round to. But since 88 is closer to 90 than it is to 80, it makes sense to round to 90. 86 or 87, likewise, will be closer to 90 than to 80. But 81, 82, 83, and 84 are closer to 80 than to 90. So our rounding rule is designed to move us to the "nearest" multiple.

The point made by this section is that

A. our rounding rule makes sense because it gets us to the multiple of 10 (or 100, or whatever) that is NEAREST our original number,
or
B. people have different ideas about what the rounding rule should be?

220. What about when the digit to the right of the rounding digit is 5? For example, suppose that we want to round 850 to the nearest 100. Our rounding rule says that 8 is the rounding digit, and we go up when the digit next to the rounding

digit is 5 or greater. So we “round up” to 900 rather than “rounding down” to 800. But is 900 the NEAREST multiple of 100? Let’s see how far 850 is from 900 and 800. 900-850 is 50. So 850 is 50 away from 900. 850-800 is 50. So 850 is also 50 away from 800. 850 is exactly in the middle between 800 and 900! So when we round 850 to 900, we’re not going exactly to the “nearest” hundred, but for the one that is tied for nearest!

The main idea is that

A. when the digit next to the rounding digit is 5, sometimes the number is right in the middle between two choices of where to round to,
or
B. rounding to the nearest 1000 means finding the number like 1000, 2000, 3000, and so forth that your original number is closest to?

221. Here’s a good brain teaser. Suppose you are the one making up the rounding rule. You have to decide: what do we do with numbers like 850? 850 is equally close to 900 and 800. How should we make the rule: to round up to 900, or round down to 800? What would be the main advantage of making a rule to round up rather than round down? Think about this for a while if you want before I tell you the answer.

Here’s the main advantage of rounding up when the digit to the right of the rounding digit is 5. It’s that if ANY digit to the right of that 5 is anything other than zero, then rounding up really does get us to the NEAREST multiple.

For example: if we’re rounding 851, then 900 is 49 away whereas 800 is 51 away – 900 is the nearest multiple of 100. The same is true for 852, 853, and so on. For all those numbers, 900 is nearest. It’s only 850 for which we have a tie. So if we make our rule to round up with 5 rather than down, we’ll be getting to the number that is really nearest most of time.

The purpose of this section is

A. to explain why when we multiply lots of numbers together, we need to round because the answer comes to too many digits for our calculators to hold in memory,
or

B. to explain why it makes sense to go up rather than down when the digit to the right of the rounding digit is 5?

222. We began this chapter with working toward the goal of seeing whether answers we get "make sense." We said that when we estimate answers, we round, do a simpler operation with the rounded numbers, and also figure out how many zeroes go onto our answer. We've talked about rounding. Let's think about how many zeroes are in our answer.

Suppose the problem is 1,249 x 372. Suppose someone gets an answer that is about 41,000. Could this answer be right? To check, let's round 1,249 to the nearest thousand, and get 1000. Let's round 372 to the nearest hundred, and get 400.

Now we want to multiply 400 x 1000. We use the rule we talked about earlier, where to multiply by a power of 10, you put the number of zeroes in that power of 10 on the right end of the number you're multiplying. So when we put 3 zeroes on the end of 400, we get 400,000. So the answer that's close to 41,000 looks as if it's a mistake, and it should be checked.

This section

A. gave more rules for rounding,
or
B. started to discuss the question of how many zeroes should be on the answer we estimate?

223. Suppose we aren't so lucky as to be multiplying by a power of 10 itself. Suppose that for example we want to multiply 400 x 2000. We can still use our power of 10 rule! We think of 2000 as 2 x 1000. 400 x 2000 becomes 400 x 2 x 1000. We multiply 400 by 2, and get 800, then add 3 zeroes, and get 800,000.

If you do enough problems like this, you find out that the following rule holds: When you are multiplying two numbers that end in zeroes, you can first just take all the zeroes off, and multiply, then replace the total number of zeroes you took off both numbers! It makes the multiplication problem lots simpler.

This section

A. stated a rule for how to multiply numbers that have zeroes at the right end,

or
B. explained what is meant by powers of ten?

224. Here’s an example of the rule we just gave. Suppose we are multiplying 8,000 by 20,000. Let’s first take the zeroes off. We take 3 zeroes off the 8,000, giving 8, and 4 zeroes off the 20,000, leaving 2. We’ve taken 4+3=7 zeroes off altogether. So now we multiply 8 by 2 and get 16, and then when we put the 7 zeroes back, we get 160,000,000 or 160 million.

This section

A. gave an example of figuring out how many zeroes should go on the answer to a multiplication problem by taking them off the factors and putting the same total number back onto the product,
or
B. gave a rule about how big numbers should be when they are answers to addition problems?

225. What about when we are dividing? A simple rule is that we can just mark out an equal number of zeroes in the divisor and the dividend without changing the answer. For example, suppose we are dividing 20,000 by 4,000. Let’s get rid of three zeroes on the right end of each number. Our problem becomes 20 divided by 4. This is lots easier! The anwer is 5, and we’re done!

The purpose of this section was to

A. tell a rule that helps figure out how many zeroes to have in division problems with rounded numbers that have zeroes on the end,
or
B. explain why you add the number of zeroes you took off each of the two numbers, and put the total number back onto the product, when doing a multiplication problem?

226. Why can you mark out the same number of zeroes in divisor and dividend when you are dividing? This is something we will discuss in more detail later. It’s the same principle that lets you divide the numerator and denominator of a fraction by the same number to get an equivalent fraction, for example with

4/6=2/3. It's the same principle that lets you say that 3/10 is the same as 30/100 or 300/1000. The principle is that when two numbers are divided by each other, you can multiply or divide both numbers by the same number without changing the quotient.

Let's illustrate that principle. 24 / 6 is 4. Let's divide 24 and 6 by the same number, and see if we get the same answer. Let's divide both of them by 2. When we do that, we get 12 / 3. We get the same answer, which is 4! How about 27 / 9 = 3. Let's divide both 27 (the dividend) and 9 (the divisor) by three. We get a new problem, 9 / 3. But the answer remains the same, namely 3! For one more example: 300 / 60 is 5. If we divide both 300 and 60 by 10, we get 30 / 6, which also gives 5.

Removing a zero from a number is the same as dividing by 10. So each time we remove a zero from both divisor and dividend, we are simply dividing both of them by 10, and this doesn't change the quotient.

The purpose of this section was to

A. tell you how to cancel when multiplying fractions,
or
B. explain why, when you are estimating the answer to a division problem, you can remove the same number of zeroes from divisor and dividend without changing the quotient?

227. We began this chapter with a question: Suppose that you divide 12,394 by 435, and you come out with 2,849. Does that answer "make sense?" Let's use the principles of this chapter to answer that question. Let's round 12,394 to 12,000. Let's round 435 to 400. So now our problem is 12,000 divided by 400. Let's get rid of 2 zeroes in divisor and dividend. This makes our problem 120 divided by 4. This is an easy division problem; we get 30. We still don't know what the exact answer to our original problem would have been but we know it would be in the general neighborhood of 30. So the answer of 2,849 does not make sense.

This section

A. calculated the exact answer to the division problem 12,394 divided by 435,
or

B. gave an example of how you use rounding and estimation to decide whether a certain answer to that division problem makes sense?

Chapter 13: Fractions

The numbers that are between whole numbers

228. So far in this book we have been talking about whole numbers. If we imagine a number line, each whole number is represented by a tiny point on the line right above the whole number. But what about all the points in between those? We need to be able to represent those, too.

We find this out whenever we set out to measure things with a measuring stick. Not every distance comes out to be a whole number of inches, or centimeters, or whatever unit we are using! Very often it is useful to refer to parts of numbers. That's what fractions are – in fact, the word "fraction" means "part."

The purpose of this section was to

A. teach how to do important operations with fractions,
or
B. introduce the notion of fractions as parts of whole numbers?

We approach fractions through the idea of division

229. The idea of division, which we are already familiar with, is our avenue into the study of fractions. Let's think about what we do when we divide whole numbers. We parcel them out into equal portions, and see how big each portion is. Suppose that we have a set of 10 things, and we want to divide them into 10 equal parts. How many are in each part? This is a simple division problem: 10/10 =1. There's one thing in each of the 10 equal parts. If we have 6 things to divide into 3 equal parts, we do 6/3 and get 2.

Now, however, suppose that we have 1 thing that we want to divide into 10 equal parts. We use the same sign that we used for division, and the problem is 1/10. When we write the problem, for this one, we also have our answer! We read 1/10 not only as "one divided by ten," but as "one-tenth." How big is each part? One-tenth of the whole thing!

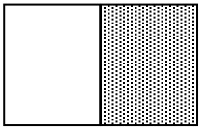
Let's imagine that it's an apple that we want to divide into 10 equal parts. We need a knife, don't we? And we also would find it difficult to get the 10 parts equal, but that's not our concern here. (But that's why in our drawings below, when we illustrate fractions, we'll use rectangles instead of apples!) When we've sliced the one apple into 10 equal parts, each part is 1/10, meaning both "one divided by ten," and "one-tenth."

The main idea of this section is that

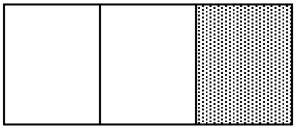
A. a fraction like one-tenth means how big each part is when you divide one thing into ten equal parts,
or
B. tenths are the portions that centimeters are divided into when you use a centimeter ruler?

The meaning of a unit fraction

230. A unit fraction is how large each one piece is when we divide one thing into a certain number of parts. (The word "unit" signals that we're talking about only one of those pieces.) Below is a rectangle. It is divided into two equal parts. One of those parts is shaded, and the other is not. We say that one half of the rectangle is shaded. We write "one half" like this: ½ or $\frac{1}{2}$. ½ is a fraction. The number on the bottom of the fraction tells how many equal parts there are. If we want to know what fraction is shaded, the number on the top tells how many parts are shaded. So in the rectangle below, there are two equal parts, and one of them is shaded. So the fraction shaded is ½.



Now let's look at a different rectangle.



What fraction of this rectangle is shaded? It's divided into 3 equal parts, and 1 part is shaded. For a fraction, the number on the bottom is the number of equal parts, and the number on top is the number of shaded parts. So the answer is 1/3 or $\frac{1}{3}$.

One of the purposes of this section was to

A give some pictures of "unit fractions," or the size of one part when one thing is divided into a certain number of parts,
or
B. to tell how to add fractions?

Non-unit fractions

231. Once we have a part of something defined, like a tenth or a third or a fifth, we can count those parts with our ordinary whole numbers.

In the rectangle above, there are five equal parts. Each part is one fifth. This time, two of them are shaded. So, what fraction of the whole rectangle is shaded? Two fifths. We write this as 2/5. The bottom number is the number of equal parts that the rectangle is divided into. The top number is the number of parts that are shaded. (The number is on top if we draw a horizontal line between the two numbers. If we use a slash mark to separate them, as in 2/5, the "top" number is the one on the left and the "bottom" number is the one on the right.)

2/5 also means "two divided by five." Does it make sense that if we divide two things by five, we'd have twice as much as if we only divided one thing by five? Twice as many as 1/5 is 2/5.

The purpose of this section was to

A. explain why we sometimes turn fractions upside-down,
or
B. define and illustrate non-unit fractions like 7/10 or ¾?

The words numerator and denominator

232. We just mentioned that sometimes the parts of a fraction are written so that they are beside each other rather than with one on top of the other. Perhaps that's why someone decided that the words *top* and *bottom* weren't good enough in talking about the parts of a fraction. That person made up some longer words: *numerator* and *denominator*.. In the fraction 2/3, two is the numerator and three is the denominator. The denominator tells how many equal parts the thing is divided into, and the numerator tells how many of those parts we have.

If you want a way to remember that the numerator is up and the denominator is down, maybe it will help that *denominator* and *down* both begin with the letter d. The word *numerator* also has a *u* in it, just as the word *up* does.

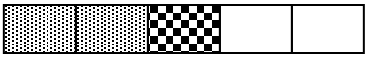
The point of this section was

A. to show that the numerator and the denominator of a fraction make a ratio with each other,
or
B. to define the words numerator and denominator as the top and bottom part of the fraction?

Adding like fractions

233. What are "like fractions?" They are fractions that have the same denominator. 2/5 and 1/5 are examples of like fractions. How do you add those? If one person has 2/5 of an object, and another person has 1/5, how much of it do they have altogether? It makes a lot of sense that each of the pieces they have is a fifth, and they have three altogether, so they have three fifths. Below is a picture of this. The object is divided into fifths. One person has the two fifths on the left (the ones shaded with dots) and the second person gets the fifth to the right of

those (the one shaded with a checkerboard pattern). How many fifths do they get altogether? 3/5.



The rule for adding like fractions is a simple one. You just leave the denominator the same, and add the numerators. So 3/7 plus 2/7 = 5/7. Three eighths plus four eighths equals seven eighths. And so on!

The lesson of this section was that

A. you have to make fractions into like fractions before you can add them,
or
B. to add like fractions, you add their numerators and leave the denominator as it is?

Subtracting like fractions

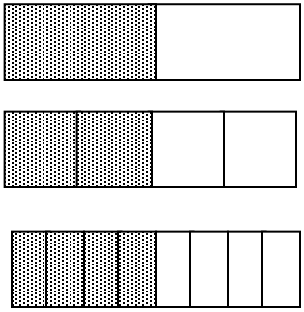
234. If one fifth plus two fifths equal three fifths, we would certainly expect that three fifths minus two fifths would equal one fifth. And that's exactly the way it is. To subtract like fractions, we simply subtract the numerators, and leave the denominator the same. So 6/7-2/7=4/7. 7/10-4/10=3/10. And so on.

To summarize this section,

A. to subtract like fractions, just subtract the numerators, and leave the denominator as it is,
or
B. if we multiply the numerator and denominator of any fraction by the same number, we get an equivalent fraction?

Equivalent fractions

235. Please look at the rectangles below. We can see that each of the three large rectangles has half of it shaded. The fraction shaded is one half.



The one on the top is divided into two equal parts, and one part is shaded. So ½ is shaded, just as our eyes told us. The one in the middle is divided into 4 equal parts, and 2 of them are shaded. So 2/4 are shaded. And the one on the bottom is divided into 8 equal parts, and 4 of them are shaded. So 4/8 are shaded.

So we started out saying that half of each rectangle was shaded, and we then figured out that 2/4 of one and 4/8 of the other are shaded. What are we to conclude from this? We must conclude that ½, 2/4, and 4/8 are all equal to each other, and so they are. We call equal fractions equivalent fractions.

The conclusion this section came to was that

A. equivalent fractions are really obtained by multiplying any fraction by another fraction equal to one,
or
B. 1/2, 2/4, and 4/8 are all equivalent fractions.

236. Suppose that we start with ½, and multiply the numerator and denominator by 2.We get 2/4, an equivalent fraction. If we multiply the numerator and denominator of ½ by 4, we get 4/8, an equivalent fraction. It turns out that if we multiply the numerator and denominator of any fraction by the same number, we will get an equivalent fraction.

We can visualize this using rectangles. Let's say we start out with the rectangle divided into a certain number of equal parts, and a certain number of those shaded, such as in the ½ rectangle above. Then we draw lines so as to divide each one of those equal parts into, let's say, 2 equal parts. Now we have 2 times as many equal parts in our whole rectangle (4), and we have 2 times as many shaded parts (2). So now we have an equivalent fraction with a numerator and a denominator each twice as big. If we had divided each small rectangle into 4 equal parts instead of 2, we would be, in effect, multiplying the numerator and denominator of ½ by 4.

The main point of this section is that

A. a big reason for changing fractions to equivalent fractions is to change unlike fractions to like fractions so we can add them,
or
B. to make a fraction equivalent to another, you multiply the numerator and the denominator of the fraction by the same number.

237. Suppose we have the fraction 2/3 and we want to change it to an equivalent fraction that is a certain number of ninths. In other words, our problem is

2/3 = ?/9

Here's the way our thought process goes. When we make equivalent fractions, we multiply numerator and denominator by the same number. What number did we multiply 3 by to get 9? We multiplied it by 3. So we also have to multiply the 2 by 3. When we do that, we get 6. So

2/3 = 6/9

and we have made the equivalent fraction with the denominator that we wanted.

This section

A. proved that if you multiply numerator and denominator by the same number, you get an equivalent fraction,
or

B. gave an example of the thought process you use when changing a fraction to an equivalent fraction with a different denominator?

Reducing fractions

238. We just talked about how you can multiply the numerator and denominator of a fraction by the same number, to get an equivalent fraction. If you think about it, this means that you can also divide the numerator and denominator by the same number, and still get an equivalent fraction. Why must this be true? Because the equal sign goes in both directions, and division is the inverse of multiplication. That's probably not clear without an example, so here is one: Let's start with 1/3, and multiply numerator and denominator by 2, to get 2/6. If 1/3=2/6, then 2/6=1/3. And the way we get from 2/6 back to 1/3 is by dividing the numerator and denominator by two. So dividing numerator and denominator by 2 gave us an equivalent fraction. Here's another example of exactly the same sort. Let's start with 3/4. Let's multiply top and bottom by 3, to get 9/12. If 3/4=9/12, then 9/12=3/4, and the way we get back from 9/12 to 3/4 is by dividing numerator and denominator by 3. Dividing numerator and denominator by the same number always gives an equivalent fraction.

The purpose of this section was

A. to explain how to reduce fractions most efficiently,
or
B. to explain why when we divide numerator and denominator of a fraction by the same number, we get an equivalent fraction?

239. Dividing numerator and denominator of a fraction by the same number to get an equivalent fraction is called *reducing* the fraction. If we keep reducing until there is no whole number which will divide evenly into both numerator and denominator, we say that we have reached the *lowest terms*, or that the fraction is *reduced to its lowest terms*. Suppose we do a problem and the answer we come out with is 18/24. We realize that we can divide numerator and denominator by 2, and we get 9/12. Then we realize we aren't done, and that we can still divide numerator and denominator by 3; we do that to get 3/4. With 3/4, there is no

longer anything we can divide evenly into numerator and denominator, so we're now reached the lowest terms of the fraction 3/4.

The main idea is

A. if you don't reduce fractions to their lowest terms, you're likely to miss test questions, despite having done everything else right,
or
B. if you keep dividing numerator and denominator by the same number until there no longer is any number by which numerator and denominator are both divisible, you've reduced a fraction to its lowest terms.

240. Why is it useful to reduce fractions to their lowest terms? We reduce fractions to make them more recognizable. Sometimes fractions that result when we solve problems look like strangers, but they are really old friends. For example, suppose we come out with an answer to a problem that is 42/56, or 39/52. Both of these, when reduced, come out to the old familiar fraction of 3/4. Also, often teachers and other people like it when there is only one right answer to a problem. If we insist that you reduce fractions, there will be only one right answer, whereas if we accept unreduced fractions, there is an infinite number of solutions to just about any problem!

A good title for this section would be

A. two good reasons to reduce fractions,
or
B. why like fractions are easier than unlike fractions?

Different expressions that mean one

241. What does 2/2 equal? We can read it as 2 divided by 2, or 2 halves. Either way, the answer is one. How about 3/3? Also equal to one. How about 24/24? The same. Any number divided by itself is one. How many sets of y things can we make out of y things? One set. This is another way of saying that y/y=1. If we divide one thing into y parts, and then put those y parts back together again, how

many things do we have? The one thing that we started out with. So any fraction with the same numerator and denominator is equal to one.

Another way of understanding that any fraction where the numerator and denominator are the same is equal to one is by our procedure of reducing fractions. With 9/9, for example, we divide numerator and denominator by 9 to get 1/1, or 1.

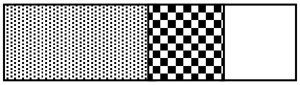
The main point is that

A. a fraction with the numerator and the denominator equal is an improper fraction,
or
B. a fraction with the numerator and the denominator equal is equal to one?

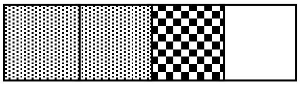
Adding unlike fractions

242. How do we add unlike fractions? The quick answer is, we don't. We change them to like fractions first, and then add the like fractions. Changing unlike fractions to like fractions is usually called "getting a common denominator."

Below is a rectangle with one-half shaded with small dots, and one-fourth shaded with a checkerboard pattern. How much of the rectangle is shaded in all? In other words, what is one-half plus one-fourth?



Let's split the one-half that is shaded with small dots into two equal pieces, like this:



What have we done? We've changed the one-half into the equivalent fraction of two fourths. We did this by splitting our drawing, but we could have also multiplied the numerator and denominator of one half by two. Now we have two fourths shaded with small dots, and one-fourth shaded with a checkerboard pattern. We have three fourths that are shaded altogether, because two fourths plus one fourth is three fourths.

This section said that

A. to add unlike fractions, you change the fractions to like fractions and then add them. You change them by multiplying numerator and denominator by the same number.
or
B. to multiply two fractions, you just multiply the numerators by each other and the denominators by each other.

243. In the problem we just did, we could turn the fractions into like fractions by changing just one of them. But sometimes we have to change both of them to equivalent fractions. Suppose our problem is 2/3 + 1/4. If we want to change thirds to fourths, we would start out by asking, "What do we multiply 3 by to get 4?" But 4 is not a multiple of 3, so going that route makes things more complicated rather than easier.

But then an idea occurs: 12 is a multiple of both 3 and 4. Why don't we change them both to twelfths?

$$\begin{array}{rcl} 2/3 & = & ?/12 \\ +\ 1/4 & = & \underline{?/12} \end{array}$$

It helps if we write the fractions to be added vertically, and make an equals sign just to the right of each of them. Then we put 12 in the denominator of each of the new fractions.

Then our thought process is like this: we multiply three by four to get twelve, so we'll multiply two by four to get eight. So two thirds equals eight twelfths. Then we do the same thing with the one fourth: we think, we multiply four by three to get twelve, so we multiply the one by three to get three. So one fourth equals three twelfths. Then we just add the like fractions 8/12 and 3/12 to get 11/12.

$$\begin{array}{rcr} 2/3 & = & 8/12 \\ +\ 1/4 & = & \underline{3/12} \\ & & 11/12 \end{array}$$

This section illustrated

A. changing improper fractions to mixed numbers,
or
B. adding unlike fractions by changing them to equivalent like fractions?

The easiest common denominator

244. When we wanted a number that was a multiple of both 3 and 4, to use as a common denominator when adding thirds and fourths, how did we know that 12 would work? Because three times four is twelve. We know twelve is a multiple of three because we multiplied something by three to get it. We know twelve is a multiple of four because we also multiplied something by four to get it! Any number that is the product of two numbers has to be a multiple of both of them.

So if you’re looking for a common denominator for two numbers, one that has been called the “easiest common denominator” or the “easiest common multiple” of the two numbers is what you get just by multiplying the numbers together.

The point is that

A. when getting common denominators, sometimes you save yourself work by picking the lowest one there is,
or
B. the “easiest common multiple” of two numbers is found by just multiplying the two numbers together?

The lowest or least common denominator

245. Suppose we are adding 1/12 + 1/24. If we use the "easiest common denominator," which is the product of the two numbers, we come up with 288! We then would change our problem to 24/288 + 12/288= 36/288. Then we would reduce that fraction by dividing numerator and denominator by 12. We would get 3/24. We would get the right answer, but we would have to deal with some time-consuming computations.

But we could save ourselves some effort by picking a lower, or in fact the lowest, common denominator, in this case 24! Then our problem becomes: 2/24 + 1/24= 3/24.

The main point of this example is that

A. using the "easiest common denominator" will obtain for you the right answer when adding fractions,
or
B. Sometimes the "easiest common denominator" gets us into more work than picking the "lowest common denominator."

246. How do you find the lowest common denominator, when you are adding fractions? This question is the same as asking, how do you find the lowest common multiple of two numbers? There's a complicated way, involving breaking both numbers down into their prime factors. But an easier way is simply to skip-count by both numbers, write down the results, and look for the first number that's in both lists. Let's imagine we're adding 1/12 + 1/18. The "easiest common multiple" would be 12 x 18, or 216. That's a little too cumbersome. Let's use the skip counting procedure:

by 12's:

12, 24, 36, 48, 60

by 18's:

18, 36

As soon as we see 36, we see that we have a number that is a multiple of both 18 and 12, since it's in both lists. So we use 36 as our common denominator.

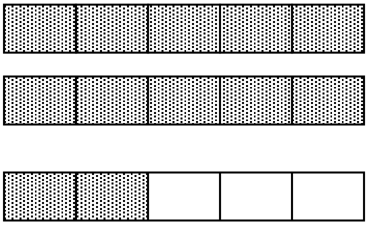
$$\begin{array}{r} 1/12 = 3/36 \\ +\underline{1/18} = \underline{2/36} \\ 5/36 \end{array}$$

The message of this section is

A. if you multiply together the prime factors of both numbers, using the highest power of each and including only once any factors common to both numbers, you get the least common multiple,
or
B. if you skip count by two numbers, making a list of the numbers you land on, the first number that's in both lists is the least common multiple?

Mixed numbers

247. Please look at the rectangles below:



In this drawing, please consider each of the large rectangles as 1 rectangle, and each of the small ones as a fifth of a large rectangle. How many large rectangles are shaded? 2 whole rectangles are shaded and 2/5 of another rectangle are shaded. So we say that 2 2/5 rectangles are shaded. 2 2/5 really means 2 plus 2/5 more. The number 2 2/5 is a mixed number. It is mixed because it has a whole number part, 2, and it also has a fractional part, 2/5.

The idea of this section is that

A. you can change a fraction into an equivalent fraction,
or
B. a mixed number is a whole number plus a fraction?

A mixed number can be expressed as an improper fraction

248. Please look again at the rectangles above, the ones that represented 2 2/5. Each rectangle is divided into fifths, including the 2 rectangles that are fully shaded. How many fifths are there in all? Let's count them: there are 5 in the first rectangle, 5 more in the second, and 2 in the last one. In all, there are 12 of them shaded. So 2 2/5 is the same as 12/5.

A fraction where the numerator is greater than or equal to the denominator is called an improper fraction; 12/5 is an example of an improper fraction. Any improper fraction can be changed to a mixed number or whole number, and any mixed number or whole number can be changed to an improper fraction.

The main idea is that

A. a decimal like 3.2 is really a special type of mixed number,
or
B. any mixed number can be expressed as an improper fraction, where the numerator is larger than the denominator?

We can turn whole numbers into improper fractions

249. Let's imagine that we have a whole number of apples: let's imagine 3 of them. Now suppose that we cut each one of them in half. How many halves do we have altogether? Each apple furnishes a set of 2 half-apples, and we have 3 such sets, so we can get the answer by multiplying: we have 2 x 3 or 6 half-apples. So our 3 apples = 6/2 apples. 6/2 is an improper fraction equal to our 3 apples.

Now let's put the apples back together (anything is possible in imagination) and this time cut them into 3 equal parts, or thirds. How many thirds do we have? Each apple supplies a set of 3 thirds, and there are 3 sets, one for each apple. So again, we can multiply to find out that there are 9/3. 9/3 is another improper fraction equal to 3 apples.

If we put the apples back together and cut them into 4 equal parts, we come out with 12/4. If we cut them into 5 parts, we come out with 15/5. If we cut them into 10 parts, we come out with 30/10. All of these are improper fractions equal to 3.

What's the pattern in these results? However many equal parts we split each apple into comes out to be the denominator of our improper fraction. And that denominator, times the number of apples (in this case three) turns out to be the numerator of the fraction.

If we extend this pattern, we find out that we can turn any whole number into an improper fraction with whatever denominator we want. We just multiply the whole number by that denominator, and that becomes the numerator of our improper fraction. The denominator we wanted is the denominator!

A summary of this section is that

A. to change any whole number to a fraction with any denominator you want, multiply the whole number by that denominator and use the resulting product as the numerator,
or
B. 3 apples is the same as six half-apples?

250. Let's look at some more examples of the procedure we just described. Suppose the whole number is 5, and we want to change it into a certain number of fourths. Each of the 5 ones has four fourths, so we have 5 x 4 or 20 fourths altogether. So our improper fraction equal to 5 is 20/4.

Suppose the whole number is 3, and we want to change it into a certain number of tenths. Each of the three ones is ten tenths, so we have thirty tenths altogether. So 3 is equal to 30/10.

This purpose of this section was to

A. furnish some more examples of turning whole numbers into improper fractions with a certain denominator,
or
B. furnish some more examples of changing improper fractions into mixed numbers?

The meaning of "firsts"

251. While we're changing whole numbers into improper fractions, let's ponder this: suppose we don't like big numbers. We want to change the whole number 4, let's say, into an improper fraction, using the smallest possible denominator. That means we want to cut each of the four imaginary apples into as few equal pieces as possible. We don't want to cut them into thirds, halves is fewer. But how about leaving them in one piece, without cutting them at all? Now each apple is left, not in two or three equal pieces, but in one equal piece! How many of those pieces do we have? We have 4, the same 4 apples that we started with! So our improper fraction is 4/1. As in all fractions, the denominator tells how many equal pieces each is divided into, and the numerator tells how many such pieces there are!

This means that any time we want to express a whole number as a fraction, all we have to do is to put that whole number in the numerator, and put a one in the denominator!

If we think of the / sign as meaning "divided by," we see that 2/1=2, 3/1=3, and so forth. Looking at it in this way just reinforces the idea that a certain number of "firsts" is just equal to that whole number.

Which of the following is a consequence of the reasoning in this section?

A. three-halves is the same as one and one-half,
or
B. seven is the same as seven firsts, or 7/1?

The procedure for changing mixed numbers to improper fractions

252. Let's remember that a while ago we looked at a picture of 2 2/5, and counted up 12/5. How could we have changed the mixed number to an improper fraction without drawing a picture and counting?

Earlier we talked about changing a whole number to an improper fraction. We found that we could change a whole number to a fraction with whatever denominator we wanted, by using as the numerator the whole number times that denominator. So to change 2 to fifths, we would recall that one is five fifths, and we have two sets of those; so 2 is 10 fifths.

So in 2 2/5, we have ten fifths coming from the two, and two more fifths coming from the 2/5. Ten plus two is twelve fifths altogether.

There's a pattern in what we did that we use whenever we change a mixed number to an improper fraction: we multiply the denominator by the whole number, and add the numerator of the fractional part. We put what we get over the denominator of the fractional part. And that gives us our improper fraction.

Let's do it with 3 ½. Two times 3 is 6, plus 1 is 7, over 2 gives us 7/2.

Let's do it with 4 2/3. Three times 4 is 12, plus 2 is 14, so we get 14/3.

A summary of this section is that

A. to change an improper fraction to a mixed number, divide the numerator by the denominator; express the remainder as a fraction by putting it over the divisor,
or
B. to change a mixed number to an improper fraction, multiply the denominator by the whole number and add the numerator of the fractional part; put the total over the denominator of the fractional part?

253. Why would anyone want to change mixed numbers to improper fractions? Mixed numbers have a real advantage over improper fractions, when it comes to communicating meaning. If I tell you that a board is 29/3 feet long, it's hard to form a mental image of how long it is. Do you realize right away that it's very similar to a board that is 68/7 feet long? But if I tell you that the first board is 9 2/3 feet long, and the second is 9 5/7 feet, you know right away that both are between 9 and 10 feet long. So why would we want to take a meaningful mixed number and turn it into a less meaningful improper fraction?

The answer is, to make certain computations easier. Multiplication and division with mixed numbers are easier if you change to an improper fraction first. If you want to multiply by a mixed number without first changing to an improper fraction, you can do it, but it's a little cumbersome; you have to use the distributive law, because a mixed number is really a sum of a whole number and a fraction. But if you want to divide by a mixed number, you really have to change it in order to figure out the answer. We'll talk more later about how you multiply and divide mixed numbers by first changing them into improper fractions.

This section told you that

A. even though mixed numbers communicate how big the numbers are, more clearly than improper fractions, improper fractions make multiplication and division much easier,
or
B. when you change from a mixed number to an improper fraction, you multiply the denominator by the whole number and add the numerator, and put the total over the denominator?

Changing improper fractions to mixed numbers

254. Suppose we have a number like 9/2. How do we change it to a mixed number? The easiest way to remember this is to recall that the sign / also means "divided by." We simply divide 9 by 2. When we do, we get 4 r 1. But the 1 that is left over also must be divided by 2, to get one half. So the answer is 4 1/2.

Here's another example. What mixed number is equal to 25/7? We divide 7 into 25, and we get 3 with 4 left over; when that 4 is also divided by 7 we get 3 4/7.

We can check our answers for these two problems by going the other direction, and changing the mixed numbers back into improper fractions. For 4 1/2: two times four is eight, plus one is nine, so the improper fraction is 9/2, the one we started with. For 3 4/7: 7 times 3 is 21, plus 4 is 25, so we get 25/7, the number we started with.

The point of this section is

A. when we multiply two fractions, we multiply the numerators by numerators and denominators by denominators,
or
B. to change an improper fraction to a mixed number, divide the numerator by the denominator, and if there is a remainder, that becomes the numerator of the fractional part of the mixed number?

Adding mixed numbers

255. Suppose we want to add 3 2/7 plus 4 3/7. We don't need to change each mixed number to an improper fraction (although you could if you wanted to; it would just result in some unnecessary work). The commutative and associative

laws of addition allow us to rearrange these numbers in our minds to become 3 + 4 + 2/7 +3/7. We can just add the whole number parts, and add the fractional parts, and we're done with this problem. We get 7 5/7.

  3 2/7
+4 3/7
  7 5/7

The main idea of this section was
A. to add mixed numbers, you just add the whole number parts, and add the fractional parts,
or
B. the associative law of addition says that instead of grouping 3 and 2/7 together in the problem above, we can group 3 and 4 together?

256. There are just a few things that can complicate adding mixed numbers. One is that the fractional parts of the numbers can have different denominators. If this is the case, we change those fractions so that they have the same denominator, or "get a common denominator," and copy over the whole number parts of the mixed number (so that we won't forget about them). Here's an example: what is 2 1/5 + 4 1/2? To solve, we just write the numbers vertically, put an equals sign to the right of each of them, change the fractional parts so the denominators are the same, and then add. Here's what happens:

  2 1/5 = 2 2/10
+ 4 1/2 = 4 5/10
          6 7/10

If our answer comes out with a fractional part that can be reduced, then we reduce it. Suppose our answer had come out 6 8/10. Then we would have reduced the fractional part by dividing top and bottom by 2, to get 6 4/5.

This section told us that

A. there can be a complication when subtracting mixed numbers, when the fractional part for the minuend is less than that for the subtrahend,
or

B. when you’re adding mixed numbers where the fractional parts are unlike, you just change the fractional parts to like fractions and then go ahead and add?

257. Another small complication comes up when adding mixed numbers, when you come out to a number that is 1 or over after adding the fractional parts. When this happens, you have to increase the whole number part of your answer. Here’s an example: what is 3 1/2 plus 2 1/2?

 3 1/2
+2 1/2
 5 2/2

So our answer is five and two-halves. But 2/2 is equal to 1. So we can add 5 and 1 to get 6 as our final answer.

Here’s a little more complex situation:

 4 2/3
+2 2/3
 6 4/3

So 6 4/3 would be a fine answer, except that 4/3 is an improper fraction. So we divide 3 into 4 to change that to a mixed number, and we get 1 1/3. Then we add the 6 to the 1 1/3 to get 7 1/3. And this is our final answer.

The main idea is that

A. when you’re adding mixed numbers, if the sum for the fractional parts comes out to be an improper fraction, you just change it to a mixed number (or whole number) and add that to the sum for the whole number parts,
or
B. when you’re adding mixed numbers, if the answer comes out with a fractional part that should be reduced, you reduce that fractional part?

Subtracting mixed numbers

258. Subtracting mixed numbers, in many cases, provides us no new challenges. We do much of what we do when adding mixed numbers, only with subtraction! We change to common denominators if we need to, and when we get the answer we reduce the fractional part if we need to. Here's an example:

$$\begin{array}{rl} & 4\ 2/3\ = 4\ 4/6 \\ - & \underline{1\ 1/6\ = 1\ 1/6} \\ & \qquad\quad 3\ \ 3/6 = 3\ 1/2 \end{array}$$

This section made the point that

A. when you read measurements from a ruler, you often use mixed numbers rather than improper fractions,
or
B. when you subtract mixed numbers, you get a common denominator and you reduce the fractional part of the answer if you need to?

259. The trickiest situation for subtracting mixed numbers comes when the fractional part for the minuend (the number you're subtracting from) is smaller than the fractional part for the subtrahend (the number you're subtracting, the one that goes on the bottom). For example:

$$\begin{array}{rl} & 3\ 1/3 \\ - & \underline{1\ 2/3} \end{array}$$

Now what do we do? We "unbundle" one of the ones in the 3, to make some more thirds to subtract from. That leaves us with only 2 ones. The one that we unbundled consists of 3/3. (One is equal to two halves, three thirds, four fourths, five fifths, etc.) The three thirds we get from unbundling get added to the one third we had to start with, resulting in 4/3. So now we've changed 3 1/3 to 2 4/3.

$$\begin{array}{rl} & 3\ 1/3 = 2\ 4/3 \\ - & \underline{1\ 2/3}= \ \underline{1\ 2/3} \end{array}$$

Now we have enough thirds in the minuend to subtract 2/3 from with no problem, and we do it.

3 1/3 = 2 4/3
- 1 2/3= 1 2/3
1 2/3

And we've got our answer.

This section made the point that

A. When subtracting mixed numbers, if the fraction in the minuend is greater than that in the subtrahend, you unbundle one of the ones in the minuend.
or
B. When subtracting mixed numbers, if the fraction in the minuend is greater than that in the subtrahend, you could if you wanted to, change both mixed numbers to improper fractions before subtracting?

Subtracting a fraction or a mixed number from a whole number

260. We just spoke about unbundling a one to make a number suitable to subtract from in certain situations. Another example of this comes up when we have a fraction or a mixed number subtracted from a whole number. Here's an example for a whole number minus a mixed number:

7
- 1 2/5

This is exactly the same sort of problem we ran into in the previous section, and the solution is the same. In fact, if we want, we can think of 7 as 7 and zero fifths, or 7 0/5. We unbundle one of the ones in 7, leaving 6. We split that one into five fifths, because we are wanting to subtract two fifths from it. So we've changed 7 into 6 5/5. After that, we just subtract in a straightforward way.

$$\begin{array}{rcr} 7\ 0/5 & = & 6\ 5/5 \\ -\ \underline{1\ 2/5} & = & \underline{1\ 2/5} \\ & & 5\ 3/5 \end{array}$$

The main point of this section is that

A. When subtracting a mixed number from a whole number, you just unbundle a one, and turn it into something like 5/5 or 4/4 or 3/3?
or
B. It doesn’t make a difference whether or not you write the whole number as something like 7 0/5 or not?

261. When you subtract a fraction from a whole number, the situation is exactly the same. You just unbundle one of the ones in the whole number. Here’s an example: suppose our problem is 9 – 2/11.

$$\begin{array}{rcr} 9 & = & 8\ 11/11 \\ -\ \underline{2/11} & = & \underline{2/11} \\ & & 8\ 9/11 \end{array}$$

An inference we can draw from the last three sections is that

A. multiplication of fractions is easier than addition and subtraction of them,
or
B. the solution is the same in each of the three situations these sections described; the main challenge for the problem-solver is to realize that the situation requires “unbundling” of a one to make a fraction large enough to subtract from.

Multiplying fractions

262. Now we’re going to work toward really understanding something that most people just memorize. Here’s what we’re going to go step by step toward understanding: that when you multiply two fractions, such as 2/3 * 3/7, you multiply the numerator by the numerator, and put that result over the denominator times the denominator. So 2/3 * 3/7 = 6/21. It’s a simple procedure. But let’s see if we can figure out why you do “top times top over bottom times bottom.”

Here's our plan. First we're going to try to understand why, when you multiply unit fractions (which are fractions with 1 in the numerator) you get 1 over the product of their denominators. For example, we're going to try to understand why 1/2 * 1/3 = 1/6. Then, we're going to try to see that non-unit fractions can be taken apart into the product of a whole number and a unit fraction. Then, we're going to use the commutative and associative laws to take one more step, to understand why the product of two non-unit fractions is the product of their numerators over the product of their denominators.

The purpose of this section was to

A. explain what the plan is for the next few sections, which will hopefully help you understand why you follow a certain procedure for multiplying fractions,
or
B. give the entire proof of why you follow a certain procedure in multiplying fractions?

Multiplying unit fractions

263. We originally defined multiplication as repeated addition. This works well for whole numbers. 2 x 3 = 2 + 2 + 2, or two put down 3 times and added.

But what do we do with a problem like 1/2 x 1/3? How are we going to add one half, one third of a time? We have to figure out some other way to make sense of the products of unit fractions.

Let's think about the product, 1 x 1/3 = 1/3. (This is of course true because any number multiplied by one is the number itself.) What happened to the 1 when it was multiplied by 1/3? The answer, 1/3, was what you get when you cut 1 into 3 equal pieces. So multiplying 1 by 1/3 was the same as dividing 1 by 3. And that makes sense, seeing that our / sign can also be read "divided by."

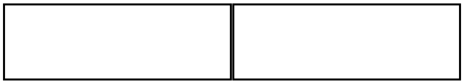
Taking any number and multiplying it by 1/3 is the same as dividing that number by 3. Or taking any number and multiplying it by 1/2 is the same as dividing the number by 2. And so on. Multiplying any number by 1/n is the same as dividing the number by n. This rule gets us part of the way toward understanding what something like 1/3 x 1/2 is.

The main rule this section tried to explain was that

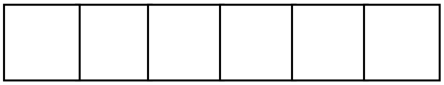
A. multiplication by 1/n is the same as dividing by n,
or
B. multiplication is repeated addition?

264. Let’s see if we can use pictures to take a guess at what 1/2 x 1/3 is.

Here’s a rectangle that’s been divided into halves.



To multiply 1/2 by 1/3, we split 1/2 into 3 equal pieces. So now let’s draw lines, dividing each of those halves into three equal pieces. We get the following:



What have we done, by dividing each of the halves into thirds? We’ve divided the rectangle into sixths! If we were to take ONE third of ONE of the halves, we’d have one sixth!

We could also do the same thing with all other unit fractions. 1/2 x 1/4 = 1/8. 1/3 x 1/4= 1/12.

The idea we come to is that you multiply unit fractions by putting one over the product of their denominators!

Another example of the rule this section states is that

A. 2 * 1/3 = 2/3
or
B. 1/5 * 1/4 = 1/20?

265. Let’s just check to make sure that it makes sense that 1/2 * 1/3 = 1/6. Our plan is as follows. Let’s take a number, such as 6, which should come out to 1 when we multiply it by 1/6. Let’s see if 6 also comes out to 1 when we multiply it

by (1/2 * 1/3). If multiplying 6 by (1/2 * 1/3) comes out to the same thing that we get when we multiply 6 by 1/6, we'll figure that 1/2 * 1/3 is the same as 1/6. Make sense?

We know that if we divide 6 into 6 equal parts, we get 1. So 6 * 1/6 equals one. Now let's see if multiplying 6 by (1/2 * 1/3) also gives us 1.

Using the associative law, we can first do 6 * 1/2. 6 * 1/2 is the same as six divided by 2, or 3. And then 3 * 1/3 is the same as 3 divided by 3, or 1. So we do come out with one! That's just what should have happened if 1/2 * 1/3 = 1/6.

This section

A. checked to make sure it made sense that 1/2 * 1/3 = 1/6,
or
B. proved that 1/2 + 1/3 = 5/6?

Multiplying a whole number by a unit fraction

266. Now let's take on something easy. What happens when we multiply a whole number, such as 5, by a unit fraction, such as 1/6? Now we can go right back to the idea of multiplication as repeated addition.

5 * 1/6 = 1/6 + 1/6 + 1/6 + 1/6 + 1/6 = 5/6.

When we multiply a whole number by a unit fraction, we just multiply the whole number by the numerator of the fraction, which is one.

For a couple more examples: 2 * 1/7 = 2/7

8 * 1/4 = 8/4.

The rule this section stated is that

A. when you multiply a whole number by a unit fraction, you just multiply the whole number by the numerator of the unit fraction, and leave the denominator as it is.
or

B. it is possible to take apart a non-unit fraction into a whole number times a unit fraction?

267. If we can multiply a whole number times a unit fraction by putting the whole number over the denominator of the unit fraction, we can also move in the other direction. We can take any fraction apart into a whole number times a unit fraction. Here are some examples:

3/4 = 3 * 1/4

7/8 = 7 * 1/8

9/16 = 9 * 1/16.

Another example of what this section is telling us is that

A. 7+9 = 16,
or
B. 5/6 = 5 * 1/6?

Multiplying two fractions

268. Now we're ready to multiply two fractions. Let's pick 2/3 and 3/5. From what we just said in the last section,

2/3 = 2 * 1/3

and

3/5 = 3 * 1/5.

So let's write 2/3 * 3/5 using these longer expressions:

2/3 * 3/5 = (2 * 1/3) * (3 * 1/5)

Now let's use the associative and commutative laws of addition to rearrange the factors, on the right side of our equation:

2/3 * 3/5 = (2 * 3) * (1/3 * 1/5)

So we get 6 * 1/15, or 6/15.

What we have done, to multiply 2/3 by 3/5, is to multiply 2*3 and put it over 3*5!

Now we've reached the end of a chain of reasoning that was meant to help you understand why, when you multiply two fractions, you put (numerator * numerator) over (denominator * denominator)!

The main point of the last section was that

A. when you multiply fractions, you get the product of the numerators over the product of the denominators,
or
B. when you divide fractions, you invert and multiply?

269. The preceding sections have tried to help us understand why, when we multiply fractions, we just multiply top by top and bottom by bottom. Or: the answer is the product of the numerators, over the product of the denominators. You can always just memorize this if you want, or practice until it comes naturally. Anyone who does a few problems like

3/5 * 4/7 = 12/35

gets into the "top by top over bottom by bottom" habit pretty quickly.

This section made the point that

A. when you are multiplying fractions, you can go through each of the steps of reasoning we went through to understand why you do what you do,
or

B. when you are multiplying fractions, you can just get into the habit of multiplying top by top and bottom by bottom, without going through each of the "understanding why" steps in your mind?

Of *means* times

270. What is one half OF 6? 3, because 3 * 2= 6. What is 1/2 TIMES 6? By what we said earlier, 1/2 * 6 = 6/2, or 3. One half OF six is the same thing as one half TIMES six. It's the same way with any fraction and a whole number. 1/3 of 12 is the same as 1/3 times 12. 1/5 of 20 is the same as 1/5 times 20.

It's also true with any two fractions. One half OF one third is the same as one half TIMES one third.

As a general rule, when we are talking about a fraction of something, OF MEANS TIMES.

An example of the message of this section is that

A. three times three is nine,
or
B. one third of nine is three, and one third times nine is three?

Multiplying mixed numbers

271. Once you know how to multiply fractions, multiplying mixed numbers is just a simple step beyond. Suppose you want to multiply 1 1/2 by 2 1/5. You just change each mixed number to an improper fraction, and then multiply top by top and bottom by bottom as when multiplying any other fractions. So

1 1/2 * 2 1/5 = 3/2 * 11/5 = 33/10

When you are done multiplying, you often have to turn an improper fraction back into a mixed number, as in this case. When you divide 10 into 33 you get 3 3/10, and that's your answer.

The rule that this section stated was that

A. to multiply two mixed numbers, just turn each into an improper fraction and multiply as you would any other fractions (that is, top by top over bottom times bottom),
or
B. to divide fractions, invert and multiply?

Canceling

272. Before starting to talk about canceling, let's think about a certain type of problem. What's 5, times 13, divided by 13? There are two ways to do this problem: a long way and a short way. Here's the hard way: 5 times 13 gives 65. Now we divide 65 by 13 and get 5. But here's the easy way: we just realize that division is the inverse of multiplication, and dividing by 13 exactly "undoes" multiplying by 13. So we get what we started out with, namely 5. Another way to think about this easy way is that dividing by 13 is the same as multiplying by 1/13. So we can say (5 * 13) * 1/13, is equal, by the associative law of multiplication, to 5 * (13 * 1/13). And (13 * 1/13) is just another way of saying 1. So 5 * 1 is 5.

Here's another example of the same type of problem. What's six billion eighty-seven, times 42, divided by six billion eighty-seven? The answer is 42, because six billion eighty-seven divided by itself comes out to one.

What's another problem of this type?

A. "What's three hundred eighty-four times four million twenty, divided by six hundred eighty-seven?"
or
B. "What's four thousand ninety-two, times eight, divided by four thousand ninety-two?

273. When we multiply fractions, we often wind up with problems that are like the ones mentioned in the previous section. Let's look at an example:

3/5 * 5/8

If we multiply numerator by numerator and denominator by denominator, we get 15/40. Now we need to reduce that fraction to get it into it's final form.

We discover that 5 is a factor of both 15 and 40, so we divide numerator and denominator by 5 to get 3/8. Hmm, we multiplied 3 by 5 and then divided what we got by 5 to get back to 3. And we multipled 8 by 5 and then divided what we got by 5, to get back to 8. Seems like some unnecessary work. It feels like doing the problems of the section above, the long way.

We've talked before about reducing fractions: you can divide numerator and denominator of a fraction by the same number, winding up with an equivalent fraction.

The new idea involved in canceling is that you don't have to multiply out the factors before you divide numerator and denominator by the same number. You can reduce the fraction first before you multiply! This idea saves us a lot of work sometimes when we are multiplying or dividing fractions.

Here's an example of the savings of work that I'm talking about. When we multiply 3/5 by 5/8, and use the associative law of multiplication, we get the following:

$$\frac{3*5}{5*8}$$

But before we even multiply out 3*5 or 5*8, we can see that 5 is a factor of both numerator and denominator. So we can divide numerator and denominator by 5, before we even multiply anything. On top, 3 * 5 /5 gives us 3, and on the bottom, 5*8 /5 gives us 8. We're left with 3 on top and 8 on bottom, so our answer is 3/8.

A short way of representing this to ourselves is simply to mark out the fives that are on the top and bottom, thinking to ourselves that they "cancel out."

$$\frac{3*\cancel{5}}{\cancel{5}*8} \quad = 3/8$$

One idea this section emphasized is that

A. in canceling, you divide numerator and denominator of the answer to a fraction-multiplication problem by the same number, before you multiply out the factors, and thus save yourself a lot of work,
or

B. it's important to realize that when two numbers "cancel out" in such fraction-multiplication problems, the result is not zero, but one?

274. Sometimes the factor we can divide numerator and denominator by is not sitting out in plain sight, as it was in the previous problem. Sometimes we have to think about what common factors are in the numbers at hand, just as when we are reducing fractions. Here's an example:

$$\frac{1}{9} * \frac{6}{7}$$

We realize that we can divide both 6 and 9 by a common factor, 3. We represent such divisions as follows:

$$\frac{1}{\underset{3}{\cancel{9}}} * \frac{\overset{2}{\cancel{6}}}{7} \qquad = 2/21$$

This section's main idea was that

A. you can save work by canceling when multiplying fractions,
or
B. sometimes when multiplying fractions there's a common factor that you have to find, that you can cancel, even though it's not sitting visibly in a numerator and a denominator?

Reciprocals and inverting

275. We've spoken several times about how, for example, multiplying by 1/3 is the same as dividing by 3. The numbers 3 and 1/3 have a special relationship. Let's define a couple of words we haven't used yet: reciprocals and inverting. To invert something (particularly, a fraction) is to turn it upside-down. And what we've got, after we invert a fraction, is its reciprocal.

For example: if we invert 2/3, we get 3/2. 3/2 is the reciprocal of 2/3. If we want to use a little more precise language than "turning it upside down," we can say that inverting a fraction means switching the positions of the numerator and the denominator.

How do we invert a whole number, like 3? We can first think that 3 is the same as 3/1. Then it's more apparent that the reciprocal of 3/1 is 1/3. The reciprocal of 1/3 is 3/1, or 3.

The key idea in this section is that

A. any whole number can be thought of as a fraction with a denominator of one,
or
B. inverting a fraction means making the numerator and denominator swap places, and what you get after inverting is called the reciprocal?

Dividing fractions: invert the divisor and multiply

276. Let's think again about how dividing by 3/1 is the same as multiplying by 1/3. Dividing by the number 3 was the same as multiplying by its reciprocal. It turns out that this is the case for any number. This fact allows us to turn any problem of dividing by a fraction into a multiplication problem! To divide by a fraction, we just invert and multiply!

For example, to divide 2/3 by 3/4, we change the problem to 2/3 * 4/3, and get 8/9.

Notice that the number we invert is the number we're dividing by, or the divisor. We leave the dividend just as it is.

With just the key idea of this section, we can do division by fractions! Everything else we need to know has already been discussed in the section on multiplication of fractions!

The key idea in this section was that

A. if you take the reciprocal of a reciprocal, you get the number you started out with,
or
B. to divide by a fraction, you invert the divisor and multiply it by the dividend?

Dividing by mixed numbers

277. The one key idea that we needed for dividing by fractions (invert and multiply) is really all we need for division by mixed numbers, also. How do we invert a number like 2 2/5? We change it to an improper fraction first, and then invert it. So to divide 1 by 2 2/5, we change 2 2/5 to 12/5. Then we invert and multiply, and our answer is 5/12.

The main idea of this section is that

A. to divide by a mixed number, you change to an improper fraction first, then invert and multiply as with any other fraction?
or
B. a mixed number is a sum of a whole number and a fractional part?

The reciprocal as one divided by the number

278. Now let's do a few problems of division by fractions, all involving 1 as the dividend. How about 1 divided by 3/4? To divide, we invert the 3/4 and multiply. We get 1 * 4/3, or 4/3. How about 1 divided by 8/7? We invert 8/7 and multiply, and get 7/8. How about 1 divided by 1/3? We invert the 1/3 and multiply, and get 3/1, or 3.

It looks as though whenever we divide 1 by a certain fraction, we get the reciprocal of that fraction, don't we? This is always the case. In fact, this is how the reciprocal is usually defined. 1 divided by any number gives the reciprocal of the number. Or, using literal numbers, the reciprocal of x is 1/x.

The idea of this section is that

A. to divide a number by any divisor, you can invert the divisor and multiply,
or
B. the reciprocal of a number is one divided by the number?

279. Here's sort of a backwards way of saying the same thing we just said. What does any number, multiplied by its reciprocal equal? The answer always comes out the same. Let's try 2/3 * 3/2. The twos cancel, and the threes cancel, leaving ones (not zeroes!). So the answer is 1. How about 3/1 * 1/3? The answer is 1

again. And it is always the case when any number is multiplied by its reciprocal, just as it is always the case when any number is divided by itself.

This section's main idea was that

A. a number times its reciprocal equals one,
or
B. the reciprocal is its own inverse function?

The number that has no reciprocal

280. When we say that "any number multiplied by its reciprocal is one," there's one number that gets excluded because it doesn't have a reciprocal. And that number is zero. The reciprocal of zero would be 1/0, but division by 0 is not meaningful. So zero has no reciprocal. If it did have a reciprocal, then that reciprocal multiplied by 0 would have to equal 1. But there's no number when multiplied by 0 that comes out to 1.

The main idea is that

A. the reciprocal of a number is defined as one divided by the number,
or
B. zero has no reciprocal?

The reciprocal is its own inverse function

281. An inverse function is one that gets us back to the starting place, or undoes what the first function does. The inverse function for "plus 5" is "minus five." What's the inverse function for the reciprocal? For example, we start with 2/3. We take the reciprocal of it, and get 3/2. What function will get us back to 2/3, from 3/2? The reciprocal itself! We take the reciprocal of 3/2 and we get back to 2/3.

Lots of calculators have a reciprocal key, although it's usually labeled 1/x. If you punch in any number (other than 0) and hit the 1/x key repeatedly, you will toggle back and forth between the original number and its reciprocal. This is a good demonstration that the reciprocal is its own inverse function.

The main idea of this section is that

A. the inverse function of plus 5 is minus 5,
or
B. the inverse function of “the reciprocal of x” is “the reciprocal of x.”

Chapter 14: Decimals

Decimals are fractions and mixed numbers, expressed in a way that extends the system we use for whole numbers

282. Ordinary fractions can be cumbersome to add, subtract, multiply, and divide. That's why someone got a very good idea, which was to use basically the same system that we use to write whole numbers, to write fractional parts of numbers as well.

Let's think about our system for whole numbers, and think once again about the "expanded form" we spoke of earlier. The number 1,234, for example, means

1 times a thousand
plus 2 times a hundred
plus 3 times ten
plus 4 times one.

As we go from left to right, each place has a value that is one-tenth of the previous one. A hundred is a tenth of a thousand, ten is a tenth of a hundred, and one is a tenth of ten.

The bright idea someone got was to put down some sort of separating mark, and just keep going, to make more places, each of which is a tenth of the previous one. (In the U.S. a dot, or decimal point, is used as a separating mark; in some other parts of the world a comma is used.)

So for the number 1,234.5678, the expanded form continues, after what we've written above, to be,

plus 5 times a tenth
plus 6 times a hundredth
plus 7 times a thousandth
plus 8 times a ten-thousandth.

The main idea of this section is that

A. the decimal system for writing parts of numbers extends the same system we use to write whole numbers, where each “place” is one-tenth the value of the place to the left of it,
or
B. the decimal number 0.8 is the same as eight tenths or eighty hundredths?

283. So decimals are just another way of writing ordinary fractions, so long as those fractions have denominators of tenths, hundredths, thousandths, or so on. So 0.3 is the same as 3/10. 0.03 is the same as 3/100. 0.003 is 3 thousandths, and 0.0003 is 3 ten-thousandths. (We could have written each of those decimals without the 0 in the ones’ place, as for example .3, .03, and .003. Especially for .3, however, the 0 helps those of us with fading eyesight to notice that there’s a decimal point there.)

This section

A. gave examples of decimal fractions that are equal to ordinary fractions,
or
B. showed how to change ordinary fractions to decimal fractions?

Reading decimals

284. How do we read a decimal like 0.27? We could call it two tenths plus seven hundredths. But if we take the fraction 2/10, and multiply numerator and denominator by 10, we find that 2/10 = 20/100. So we have 20 hundredths and 7 hundredths more, for a total of 27 hundredths. So we can read 0.27 as 27 hundredths.

If you go through this same reasoning for other decimals like 0.033, 0.276, and so forth, you find that there’s a rule that works every time: you can just read off the number after the decimal point, as though it were a whole number, and then express it with a denominator that’s the smallest place in the number. For example, with 0.27, we can just read off 27, and then, because the smallest decimal place in 0.27 is hundredths, we say 27 hundredths.

Let’s do the same thing for 0.033. We just read off the number after the decimal place as if it were a whole number, 33. (The zero on the left doesn’t change how we read 33.) The smallest, or rightmost, decimal place in the number is thousandths, so we read the number as 33 thousandths. For 0.276 we read it as two-hundred seventy-six thousandths.

This section

A. explained how to do expanded forms of decimals
or
B. explained how to read decimal fractions with more than one nonzero digit?

Adding zeros on the right of a decimal fraction doesn't change the value

285. Let's take the decimal 0.4, and add a zero on the right end, to make it 0.40. Have we changed the value of the decimal?

Using the rule stated in the section above, we read the first fraction as 4 tenths, and we read the second one as 40 hundredths.

If we were to take 4/10 and multiply numerator and denominator by 10, we would get 40/100. But multiplying numerator and denominator by the same number doesn't change the value of the fraction. So 4/10 and 40/100 are equivalent fractions.

If we had added two zeroes, we would have gotten 0.400, or 400/1000. And this, too, is an equivalent fraction to 4/10.

It works out every time, that you can add zeroes on the right end of a decimal fraction without changing the value of the number you're working with.

What's a consequence of the rule given in this section?

A. if we change 0.8 to 0.08, we do change the value of the decimal fraction,
or
B. if we change 0.8 to 0.80, we don't change the value of the decimal fraction?

286. If we can add zeroes to the right of a decimal fraction, we can also put down a decimal point as a separator and add zeroes to any whole number. So 9 is equal to 9.0, or 9.00, or 9.000, or so forth. It's nice to know that we can imagine any whole number with a decimal point after it, or even put a decimal point after it.

A consequence of the rule in this section is that

A. the number 482 is the same as 482.000 and also the same as 482. ,
or
B. the number 20 is the same as 40/2?

Multiplying decimals by 10, 100, or 1000 or other powers of ten

287. One of the nice things about decimals is how easy it is to multiply them by 10, 100, 1000, or other "powers of ten." Here's the rule: to multiply by 10, move the decimal point one place to the right. To multiply by 100, move it two places to the right; to multiply by 1000, move it three places to the right. And so on if you want to multiply by 10,000 or larger.

For example: 4.046 * 10 = 40.46
4.046 * 100 = 404.6
4.046 * 1000 = 4046.

A consequence of the rule stated in this section is that

A. 364=364.0
or
B. 367.021 * 1000= 367,021.

288. Let's think about why this works. Let's look at an example, and think about why 0.03 * 10 = 0.3. 0.03 means 3 hundredths. If we multiply

3/100 * 10/1

we can cancel with 10 and 100 and we come out with 3 tenths. And when we write 3 tenths as a decimal, 0.3, we find we've just moved the decimal point one place to the right.

The same process would hold for each digit in a decimal with several digits. Each time we multiplied any digit's value by 10, we'd get the value of the place just to the left of it – and why? Because our number system is set up so that each place is ten times that to the right of it, and one tenth of that of the left of it.

If we have a decimal with several digits, for example 0.237, multiplying that number by 10 entails multiplying the value that every digit contributes by 10.

Why? Because of the distributive law. Remember that if we want to multiply 10(2+3) we could do 10*2 + 10*3? The distributive law means that if we want to multiply

10 (0.237)= 10 (2 tenths + 3 hundredths +7 thousandths)

then we multiply each of the three added terms by 10:

10(0.237)= 10 * 2/10 + 10 *3/100 + 10* 7/1000

which comes out to

2 ones plus 3 tenths plus 7 hundredths or 2.37. And this is what we would get by simply shifting the decimal point one place to the right!

This section

A. stated that if you want to divide by 10, you move the decimal point one place to the left,
or
B. tried to explain why when you multiply a decimal by 10, you move the decimal point one place to the right?

The rule about moving the decimal point has the same result as a previous rule

289. Do you remember a previous rule that we stated about multiplying whole numbers by 10, 100, 1000, or other "powers of ten?" The rule was that you attach one zero to multiply by 10, two zeroes to multiply by 100, and so forth.

But with our rule about moving the decimal point to the right, and with knowing that we can tack on zeroes after the decimal point for any whole number whenever we want, we can do the same thing. Let's say we want to multiply 89 by 10. We just add a zero after the decimal point, so that we are multiplying 89.0 by 10. Then we shift the decimal point one place to the right, to get 890. We've done the same thing as adding a 0 to the right end of 89!

So the rule about multiplying whole numbers by powers of ten by adding zeroes is just a "special case" of our rule about multiplying any decimal number by powers of ten by shifting the decimal point to the right!

The main point of this section is that

A. to multiply any whole number by 10, 100, or 1000, add on 1, 2, or 3 zeroes, respectively, to the right end of that whole number,
or
B. the previous rule about multiplying by 10, 100, or 1000 by adding zeroes is just a special case of the rule of multiplying by shifting the decimal point to the right?

Dividing by 10, 100, 1000, or other powers of 10

290. Division by 10 is the inverse function of multiplication by 10. So if moving the decimal point one place to the right multiplies a number by 10, then we'd expect moving the decimal point one place back to the left, which would get it back to the starting point, would have to divide it by 10. And so it is. To divide any decimal by 10, move the decimal point one place to the left; to divide by 100, move two places left; to divide by 1000, move three places left; and so forth.

A consequence of the rule in this section is that

A. 14.07 divided by 100 is 0.1407,
or
B. 14.07 times 100 is 1,407?

Adding decimals

291. There are just three things to keep in mind when adding decimals.

First, as with whole numbers, line up the digits that have the same place value underneath each other in neat columns. One way to do this is to line up the decimal points of the numbers in a neat column.

Second, if there is an unequal number of digits to the right of the decimal point among the various addends, you can add 0's to the right if you wish to, so that all addends have the same number of decimal places. You don't have to do this, but it helps keep the digits lined up right. This is about the closest you're going to get, with decimals, to messing with a "common denominator."

Third, when you're finished adding the numbers, just as you would add whole numbers, put the decimal point in the answer directly under the decimal points in the addends.

And that's it! The fact that this section was so much shorter than the section on adding ordinary fractions is one of the major advantages of decimals!

The MAIN message of this section is that

A. when adding decimals, line up the digits under those of the same place value, add zeroes on the right if you wish to make your columns even, and insert the decimal point in the answer directly under that of the addends,
or
B. inserting zeroes on the right after any decimal number is optional?

292. Here's a quick example of adding decimals. Let's add 0.007 + 2.76 + 0.03. First we'll line them up with tenths under tenths, hundredths under hundredths, etc. just by aligning the decimal points.

$$\begin{array}{ll} & 0.007 \\ & 2.76 \\ + & \underline{0.03} \end{array}$$

Now we'll do the optional maneuver of adding zeroes to get an equal number of decimal places in the addends:

$$\begin{array}{ll} & 0.007 \\ & 2.760 \\ + & \underline{0.030} \end{array}$$

Now we just add the numbers, as if they were whole numbers:

$$\begin{array}{r} 0.007 \\ 2.760 \\ +\ \underline{0.030} \\ 2\ 797 \end{array}$$

Finally, we insert a decimal point directly under the other decimal points:

$$\begin{array}{r} 0.007 \\ 2.760 \\ +\ \underline{0.030} \\ 2.797 \end{array}$$

And we're done!

The point of this section was

A. to explain why the three guidelines for adding decimals are mathematically valid,
or
B. to give an example of using the three guidelines in adding decimals?

293. Let's take just a second to ask the question, why do we put the decimal in the answer, directly under the decimal in the addends, other than that some trustworthy source told us to do so?

The answer to this one isn't hard. When we are adding up the tenths' column in the problem, the answer we get is tenths. (If there are more than ten tenths, there's enough to make another one, which we carry over to the ones' column.) Then when we add up the ones' column, the answer we get is the number of ones in the sum. The decimal point should fall right between the tenths' column and the ones' column in the sum, just as it does for the addends.

This section

A. was an explanation of why the decimal place in a decimal-addition problem is directly under the decimals in the addends,
or
B. was an explanation of how to carry when adding?

Subtracting decimals

294. The guidelines for subtracting decimals are almost exactly the same as for adding. Line up the decimal points and like digits for the minuend and subtrahend. Put some extra zeroes on at the right to keep the rows of equal length and the columns lined up. (This step becomes a much more useful one in subtraction than in addition, especially when the subtrahend has more decimal places than the minuend!) Subtract just as you would with whole numbers, and when you're done, keep the decimal point in the difference right in its vertical line.

For example, we'll write a problem, lining up the decimal points:

$$\begin{array}{ll} & 6.2 \\ - & \underline{3.428} \end{array}$$

Now we add zeroes to the minuend, to make 3 decimal places in the minuend, just as there are in the subtrahend:

$$\begin{array}{ll} & 6.200 \\ - & \underline{3.428} \end{array}$$

Now we subtract just as we would with whole numbers:

$$\begin{array}{ll} & 6.200 \\ - & \underline{3.428} \\ & 2\ 772 \end{array}$$

And finally, we insert the decimal point right under the others, and we're done.

$$\begin{array}{ll} & 6.200 \\ - & \underline{3.428} \\ & 2.772 \end{array}$$

A summary of this section is that

A. the guidelines for subtracting decimals are almost exactly the same as those for adding decimals,
or
B. the great thing about "borrowing" in subtraction of decimals is that, unlike that with subtracting mixed numbers, it is done just as you do with whole numbers?

Multiplying decimals

295. Here's the procedure for multiplying two decimals. First, you just forget about the decimal points in the numbers, and line them up and multiply them just as if they were whole numbers. You don't worry about lining up decimal points. You just put the right end of one number under the right end of the other. So for example, if you are multiplying 3.14 x 1.1, you line them up and multiply them just as if they were 314 x 11 (For this example, you'd get 3454.)

Then, when you get the answer, you count off the total number of decimal places to the RIGHT of the decimal point in both of the factors. (In this example, there would be 3 total decimal places: 2 for 3.14, and 1 for 1.1) Then you start from the right and count off that number of decimal places in your answer, put down the decimal point, and you're done! In this example, the answer would be 3.454!

The better summary of this section is

A. To multiply decimals, first multiply as with whole numbers. Then, starting from the right, count off a number of decimal places in the product equal to the total places to the right of the decimal in both factors.
or
B. When adding decimals, you line up decimal points, but when multiplying, you don't necessarily.

296. Now let's take two sections to understand why you count off the number of decimal places in the product equal to the total decimal places in each of the factors. Let's first do a little exercise. The question is: suppose we have a product of two factors. Let's call that the first product. Then suppose we multiply each of the two factors by ten, to make two new factors. Then we multiply those factors, to make a second product. How many times greater is the second product than the first? Is it just ten times ten, or 100 times greater? Let's try an example and see.

For example: First let’s multiply 2 x 3, and get 6.

Now let’s multiply each of 2 and 3 by 10, and get 20 and 30, and multiply them together, to get 600. How many times greater is 600 than 6? 100 times greater.

Here’s how we could have known that the second product was going to be 100 times greater than the first product, without even multiplying anything out:

The second product is

(2 x 10) x (3 x 10)

Now let’s rearrange these, using the associative and commutative laws of multiplication:

(2 x 3) x (10 x 10)

(2 x 3) is the first product, and we see that it’s multiplied by (10 x 10) to make the second product.

So if we multiply each of the two factors by 10, we increase the product by 10 x 10. What if we had multiplied one factor by 10 and another by 100? Then we would increase the product by a factor of 10 x 100. Or to state our rule using literal numbers, if we multiply one factor by a, and another factor by b, we multiply the product by ab.

A summary of the rule we figured out in this section is that

A. If you multiply two decimals, you count off the number of decimal places in the product that’s the total number of decimal places in each of the two factors?
or
B. If we multiply one number by a, and a second number by b, we multiply their product by ab.

297. How do we apply the rule we just figured out, to try to understand the procedure for multiplying decimals? Let’s use our example above, 3.14 x 1.1. If we were to change 3.14 to 314, we would be multiplying it by 100 (because moving the decimal point two places to the right is multiplying by 100). If we

were to change 1.1 to 11, we would be multiplying it by 10. (That's because moving the decimal point one place to the right is multiplying by 10.) So by the rule we just figured out, we'd be multiplying the product by 1000. So the product we get, ignoring decimals, is 1000 times too great. To get it back to where it's supposed to be, we divide by 1000. And we do that by moving the decimal point 3 places to the left. It's no coincidence that those three places are the sum of the two places in 3.14 and the one place in 1.1!

So why does the number of decimal places in the product equal the sum of the number of decimal places in the factors? It's because if you ignored the decimal places in each factor, you'd be multiplying each of them by a certain power of 10; to make the answer come out right, you divide the answer by a certain power of 10; it turns out that moving the decimal point just the total number of places that were in both factors accomplishes this automatically!

The author's strategy in trying to help you understand the "total number of decimal places" rule when multiplying decimals was to

A. Imagine multiplying to change each of the factors to a whole number, and then dividing the product by the right power of 10 to get back to the right answer?
or
B. Round off each of the factors to get an idea of about what the product should be?

298. Once we've gotten our answer by counting decimal places, it's a good idea to check your answer by rounding off each factor and getting an estimate of what the answer should be. When we multiply 3.14 x 1.1, if we round each number to the nearest whole number, we get 3 x 1, or 3. Thus we know that 3.454 is sure to have the correct placement of the decimal, and that 34.54 or 0.3454 would surely be wrong.

The advice in this section was to

A. check your answer by rounding and estimating, after multiplying decimals,
or
B. make sure to add the number of decimal places in both of the factors to get the number of decimal places in the answer?

Dividing decimals by whole numbers

299. When we talk about dividing a decimal by a whole number, we mean that the dividend is a decimal and the divisor is a whole number. Or, the number under the division bracket is a decimal and the number to its left is a whole number. It can be a short problem like

$$3\overline{)1.2}$$

or a longer one like

$$36\overline{)72.144}$$

The rule is simple. You just divide as though both of the numbers were whole numbers. When you get your answer, you put the decimal point in the quotient directly above the decimal point in the dividend!

This section told you that, when dividing a decimal by a whole number,

A. you estimate the quotient by rounding, to make sure the answer makes sense, or
B. you put the decimal point in the quotient directly above the decimal point in the dividend?

300. The only tricky part about following this rule comes when your quotient doesn’t start until one or more columns to the right of the decimal point. In this case, you have to fill in a zero above each digit of the dividend until you get to the one above which your quotient is written. For example, in the division problem below, we put the decimal point for the quotient above the decimal point in the dividend. But then we have blank spaces above the two zeroes and the 3 in .0036.

$$\begin{array}{r} 9 \\ 4\overline{).0036} \end{array}$$

So we fill in those blank spaces with zeroes, and we get the right answer!

$$\begin{array}{r} .0009 \\ 4\overline{).0036} \end{array}$$

The purpose of this section was to

A. explain that you put the decimal point in the quotient directly above that in the dividend,
or
B. explain that sometimes, after putting the decimal point in the quotient directly above that in the dividend, you have to fill in zeroes as place-holders?

301. Now let's see if we can understand WHY we put the decimal point in the quotient directly above that in the dividend. First let's figure out a little rule about what happens to quotients when we multiply dividends by a certain number, and leave the divisor the same.

Let's look at the division problem, 8 / 2 = 4. (8 is the dividend, and 2 is the divisor, and 4 is the quotient.) Now let's multiply the dividend by 10, and leave the divisor the same. Our new problem is 80 / 2 = 40. When we multiplied the dividend by 10, we multiplied the quotient by 10.

It's the same no matter what numbers we use: if we multiply the dividend by a certain number, we multiply the quotient by the same number. If we want to think of dividing up a pie among 2 people: if our pie is ten times bigger, then each person gets ten times bigger a piece.

The rule that this section stated is that

A. when you divide a decimal by a whole number, you put the decimal in the quotient directly over the decimal in the dividend,
or
B. if you multiply the dividend by a certain number and leave the divisor the same, you multiply the quotient by the same number?

302. With that rule in mind, we're ready to understand why the decimal point in the quotient goes directly over that in the dividend. Let's imagine that we were to move the decimal point all the way to the right in our dividend, to make the dividend a whole number. We'd be multiplying the dividend by a certain power of 10: 10 if we move one place, 100 if we move 2 places, 1000 if we move 3 places, and so forth. So by the rule we mentioned in the section above, that would make our quotient 10 or 100 or 1000 or however many times too big. To make the quotient correct, we would move the decimal point back to the left, however many places we moved the decimal point to the right in the dividend. But when we do that, we would end up in just the same column that we started out with! And that's why all we have to do is to put the decimal point directly over that of the dividend.

The author's explanation of why you put the decimal in the quotient above the decimal in the dividend involved

A. thinking about how many times too big the answer would be if we changed the dividend to a whole number,
or
B. thinking of division as the inverse of multiplication and reasoning backwards from our rule about multiplying decimals?

303. Here's a second rule for where the decimal point goes when decimals are divided by whole numbers; you might find this one easier. You divide as if using whole numbers, and then you just make sure that there are the same number of decimal places (to the right of the decimal point) in the quotient that there are in the dividend!

Why does this rule work? Let's think about multiplying a decimal by a whole number. By our rule for multiplication, the number of decimal places in the product has to be the same as the number of places in the factor that was a decimal. We can check any division problem by multiplying the quotient times the divisor to get the dividend. Thus if the divisor is a whole number, the quotient and the dividend have to have the same number of decimal places!

This section

A. restated the rule that you place the decimal point in the quotient directly above that in the dividend,
or
B. stated and explained a rule giving the same result, which is that the quotient has to have the same number of decimal places as the dividend?

Dividing by decimals

304. Suppose you are dividing either a whole number or a decimal by a decimal. For example, the divisor is a decimal, like 1.3 or 2.67 or such. How do we divide in this case?

The answer is that we turn the divisor into a whole number, by moving the decimal point to the right. To make the answer come out the same, we move the decimal point in the dividend the same number of places to the right. Then we go ahead and divide as per the previous guidelines for dividing by whole numbers.

This section said that

A. when the divisor is a decimal, you just turn it into a whole number, and compensate by moving the decimal point of the dividend the same number of places you moved it in the divisor?
or
B. when the divisor is a decimal fraction, you end up doing the same thing you would do if you “inverted and multiplied?”

305. Why does our procedure of moving the decimal point the same number of places in divisor and dividend work? It’s the same principle we use when reducing fractions or changing fractions to equivalent ones. What goes for fractions, also goes for division, because the two are really the same thing. (8/2 means eight halves; it also means eight divided by two.) The principle is that you can multiply numerator and denominator of a fraction by the same number, and not change the value of the fraction. The same principle is that you can multiply the dividend and divisor in any division operation by the same number, and not change the quotient. Let’s look at some examples:

if 8 / 2 = 4,
then

80 /20 = 4 also,
and
800/200 = 4 also, and so on.

When you shift the decimal point in divisor and dividend the same number of places, you are simply multiplying each by the same power of ten.

The purpose of this section was to

A. explain why we use the procedure of shifting the decimal point the same number of places in divisor and dividend,
or
B. explain why, when we divide by a decimal, we could just divide as if by whole numbers and then put a number of decimal places in the quotient equal to those in the dividend minus those in the divisor?

How we can change ordinary fractions to decimals

306. Imagine someone's saying, "Decimals may be very good for tenths and hundredths and so forth, but what do we do about all the fractions that aren't tenths – like halves and fourths and thirds and sixths and twenty-firsts and so many others? The whole idea doesn't sound very practical to me."

It's a good thing that whoever thought of decimals wasn't discouraged by this sort of thinking. There's a very good answer to this question – it's that we can change any ordinary fraction to a decimal. In fact, there are two good ways of doing so.

The first way works sometimes, when 10 or 100 or 1000 or some other power of 10 is a multiple of the denominator of the fraction we're changing. In this case, we can just multiply numerator and denominator of the fraction we're changing by whatever it takes to change to a fraction with a decimal-type denominator. For example, let's look at the ordinary fraction 1/2.

1/2 = ?/10

We multiplied 2 by 5 to get 10, so we multiply the 1 by 5. We get that 1/2=5/10. And we can also write 5/10 as 0.5.

Now let’s do it with 1/4. 10 is not a multiple of 4, but 100 is. So

1/4=?/100

We multiplied 4 by 25 to get 100, so we’ll multipy the 1 by 25 to get 25. 1/4 =25/100 or 0.25.

This section

A. illustrated one way of changing ordinary fractions to decimal fractions,
or
B. illustrated one way of changing decimal fractions back to ordinary fractions?

307. Here’s a second way of changing ordinary fractions to decimal fractions. This way will always work. You simply express the whole-number numerator of the fraction as a decimal, such as 1.00 or 3.000 with however many zeroes you want. Then you just use long division to divide this decimal by the denominator. In other words, you divide the numerator by the denominator and carry the division to as many decimal places as you want. For example, with 1/2, we do

$$\begin{array}{r} .5 \\ 2\overline{)1.0} \end{array}$$

And with 3/4, we do

$$\begin{array}{r} .75 \\ 4\overline{)3.00} \\ \underline{28} \\ 20 \\ \underline{20} \\ 0 \end{array}$$

The method of changing fractions to decimals described in this section was to

A. express the numerator as a decimal and divide by the denominator,

or
B. multiply numerator and denominator by the same thing so as to get a fraction with a denominator that's a power of ten?

308. A quick way to change fractions to decimals is with a calculator; you just divide the numerator by the denominator. To change 3/4 to a decimal, you pick up your calculator and punch in 3 / 4. The display reads .75.

The main idea of this section is that

A. numerator is to dividend as denominator is to divisor,
or
B. to change a fraction to a decimal with a calculator, just punch in the numerator divided by the denominator?

Repeating decimals

309. Let's try changing the fraction 1/3 to a decimal. We write the numeral 1 with a decimal point and a bunch of zeroes after it. And we divide three into it. Our thought process goes like this: 10 divided by 3 is 3. 3 times 3 is 9. 10 minus 9 is 1. Bring down a zero, and we have 10. 10 divided by 3 is 3. 3 times 3 is 9. 10 minus 9 is 1. Bring down a zero. Hey, we're doing the same thing over and over again. The decimal that we get, as we keep dividing, is 0.3333333.... and so on. We can write a repeating decimal by putting a line over the repeating part.

This section pointed out that

A. some fractions never come out to a whole number of tenths or hundredths or thousandths or so forth, no matter how long we keep dividing, and these are called repeating decimals,
or
B. there's an interesting way to change repeating decimals to ordinary fractions?

310. Most of the time in real life calculations, we don't need numbers to be exactly accurate. And we can't get measurements such as length or weight or time exactly accurate anyway. We just need the numbers to be "accurate enough." We round off the numbers to whatever degree of accuracy we need.

This means that when we're representing 1/3 using decimals, it may be accurate enough to use .33. Or if we want it more accurate, we might use .333. If we want it even more accurate, we might use .33333. As we keep adding more digits, we keep getting closer and closer to the true value of one-third. Even with a couple of million digits, we still won't get to exactly one-third. But with three or four digits we get as accurate as we need for most purposes.

A point made by this section is that

A. no matter how many digits we use, we can't represent a repeating fraction as a decimal with complete accuracy,
or
B. the number of digits we decide to round off to is called the number of "significant figures" in our number?

Changing decimal fractions to ordinary fractions

311. Suppose you have a decimal fraction like .85. How would you change it to an ordinary fraction? We can immediately write it as 85/100. Now all we have to do is to reduce it. If we divide numerator and denominator by 5, we get 17/20. There's our ordinary fraction!

So to change any decimal fraction to an ordinary fraction, you just write the decimal with a denominator of some power of ten, and reduce.

A consequence of this section is that

A. To change 5/8 to a decimal, we divide 5.000 by 8.
or
B. To change .375 to an ordinary fraction, we write it as 375/1000 and reduce that fraction?

312. Here's a tip on changing decimal fractions to ordinary fractions. Ten is equal to 5 times 2. So all powers of ten are just products of a certain number of 2's and 5's. This means that if the number in the numerator of the fraction you're trying to reduce isn't even, and doesn't end in five, you've reduced that fraction as far as it can go.

What's a consequence of the idea in this section?

A. If we want to express .27 as an ordinary fraction, we just write 27/100, and we know right away that we can't reduce it, because 27 isn't even and doesn't end in five.
or
B. If we want to express 27/100 as a decimal, we wouldn't need to make a division bracket and divide 27 by 100; we could just immediately write 27/100 in decimal form, as .27.

313. If you're asked to change a number like 13.2 to a mixed number with an ordinary fraction, you just change the fractional part to an ordinary fraction and leave the whole number part as it is. So for 13.2, you'd first make it into 13 2/10, and then reduce the fractional part to 13 1/5.

The point of this section was

A. to change decimal numbers greater than 1 to mixed numbers, you just change the fractional part, and leave the whole number part the same,
or
B. 13.2 is the same as 13 1/5?

Rounding decimals

314. Suppose you want to round a certain decimal number, such as 3.14159265. We follow rules very similar to those of rounding integers.

You'll remember that if you want to round a number like 263 to the nearest hundred, the answer would be 300. This is because 2 is the digit in the hundred's place, and the digit to the right of it is 5 or greater, so we go up to 300 rather than down to 200. If we wanted to round 263 to the nearest 10, the answer would be 260. 6 is the digit in the ten's place, and we leave it the same rather than going up because the digit to the right, 3, is less than 5.

We use the same sorts of rules when we round to the nearest ten-thousandth or hundredth or tenth, etc. If we want to round 3.14159265 to the nearest ten-thousandth, the answer is 3.1416, because 5 was in the ten-thousandths' place, and the digit to the right was 5 or greater. If we want to round

3.14159265 to the nearest hundredth, the answer is 3.14, because 4 is in the hundredths' digit and the digit to the right of it is under 5.

A summary of the rule discussed in this section is which of the following?

A. To round a decimal number, you pick what place you want to round to and look at the digit in that place. You leave it the same if the digit to the right is under 5, and move up to the next digit if the digit to the right is 5 or greater.
or
B. When you multiply two numbers, and one is very accurate but the second is less accurate, you should round the answer to reflect the degree of accuracy in the less accurate factor.

315. Why do you round numbers? One reason is that you can only measure to a certain degree of accuracy. For example, suppose we are cutting down a tree and selling the lumber. We measure that the tree is one meter big around (that is, the circumference is one meter), and someone wants to know what will be the diameter (which is the distance across, right through the middle of the trunk). We can calculate the diameter from the circumference: we divide by a number called pi, which when rounded to the nearest hundred-millionth is 3.14159265. But our measuring procedures are not anywhere close to accurate enough to make us want to spend time entering all those digits. First of all, we can only read our tape measure to maybe one-thousandth of a meter, and second, there's even more variation depending on where exactly on the tree trunk we take our measure. So we decide that 3.14 gives us more than enough accuracy to make an estimate of the diameter of the tree.

The author in this section is trying to communicate that

A. the main reason you will ever round off numbers is because someone asks you to do so on a test,
or
B. one of the main reasons for rounding numbers is that we would waste time dealing with lots of digits when our measures are only accurate to a few digits anyway?

Chapter 15: Introduction to Solving Equations

Thinking more about variables

316. We spoke before about letting a letter like a, b, c, x, y, or z stand for a number. There are several reasons for using literal numbers (or variables). One is that we want to make a general statement, without having to spend eternity listing all the special cases of it, such as with $a+b=b+a$.

Another reason for letting a letter represent a number is that we don't know what the number is. The letter can help us represent the information we have about that number, and sometimes we can figure out what number the letter has to stand for, from the information we're given. Let's think some more about using variables in this second way.

According to this section, we will be exploring further the use of variables

A to make more general statements than are possible with regular numerals,
or
B. to let variables represent unknowns, and help us represent the information that may let us figure out what the unknown number is?

Evaluating an expression when a variable takes on a certain value

317. A representation of literal and/or ordinary numbers combined in some way is called an algebraic expression. $2x + 7$ is an algebraic expression. $3 – 14*(x/7)$ is an algebraic expression.

Very often it's useful to answer questions like, "What is $2x+7$, when x is 3?" This is called evaluating the expression $2x+7$ when x "takes on" the value 3. To answer this question, we just rewrite

$2x + 7$ substituting 3 each time we see x. We need to insert a multiplication sign to avoid confusion. We get

$2*3 + 7$, which is $6 + 7$, or 13.

Here's another example. What is (2x + 6)/4 when x is 7? We just rewrite the expression, substituting 7 for x:

(2*7 +6)/4

Then we follow our rules for order of operations. We do what's in parentheses first. Inside the parentheses, we do the multiplication first, and get

(14+6)/4

Then we finish what's in parentheses:

20/4
And finally finish with the division to get 5, our answer!

A summary of this section is,

A. to evaluate an expression when a variable takes on a certain value, we just substitute the value for the literal number and follow the rules for order of operations,
or
B. we do multiplications before addition in our order of operations?

Using variables to find unknowns, or solve equations

318. Here's an example of the use of variables to find unknowns.

Ted is a certain secret number of years old. In 5 years, Ted will be 15 years old. Can you figure out how old Ted is now?

You might have already figured out the answer, but let's go through how you would use literal numbers to solve this problem.

One intermediate goal is to write an equation that we can then solve. We want a statement that something is equal to something else.

First, let's make up a letter that stands for our unknown, which is Ted's age now. x is one of the all-time favorites for unknowns; let's pick it. If Ted is x years old now, how old will he be 5 years from now? We can think: if he were 5 years old, he'd be 5+5 or 10, five years from now. If he were 6 years old, he'd be

6+5 or 11, five years from now. So however old he is now, he'll be that number plus 5. So if he's x years old now, he'll be x+5 years old 5 years from now. We're representing something that we're told, in mathematical symbols, using a literal number.

So 5 years from now, Ted will be x+5 years old. But we're also told that 5 years from now, Ted will be 15. So the expression "15" is a second representation of "how old Ted will be 5 years from now." Since the two expressions, x+5 and 15 represent the same thing, we know they are equal to each other. So we can write the equation,

$$x+5=15.$$

This section
A. proved that x+5 is always equal to 15, just as the area of a rectangle is always equal to the length times the width,
or
B. gave an example of the thought process someone goes through in solving a word problem by writing an equation?

319. Now let's continue with the same problem. When we left off, we had let x equal Ted's age now, and we had figured out that

$$x+5=15.$$

Now if you already know what you have to add to 5 to get 15, pretend you don't for a bit. I want to introduce part of what someone called the "golden rule of equations": if you do the same thing to both sides of an equation, the equation remains true. If two expressions are equal, you can add the same number to each side, or subtract the same number from each side, and the equation will stay true. Or you can multiply or divide each side by the same number. Or you can do other more complicated operations to both sides. Let's take a second and see that this is true with an equation like

$$8=8.$$

If we add 2 to both sides of the equation, we get

$$10=10.$$

Now let's divide each side of the equation by 2: we get

5=5.

If we now subtract 4 from each side, we get

1=1,

and if we now multiply each side by 17.57, we get

17.57=17.57.

Doing this exercise makes it obvious that if we do the same thing to both sides of an equation, the numbers we get remain equal. It's less obvious when we're dealing with something like x+5=15, but still seems that it must be true.

This section

A. finished solving the equation x+5=15,
or
B. took time out from solving the equation x+5=15 to state and give examples of the "golden rule of equations," which is that if you do the same thing to both sides, the equation remains true?

320. Now let's use the "golden rule of equations" to solve the equation x+5=15. We'll do it by subtracting 5 from each side of the equation. When we do that, we get
x+5-5=15-5.
Adding 5 to x and then subtracting it again just gives us x. And 15-5 gives us 10. So if we rewrite our equation in a simplified way, we get
x=10.

And this is the solution to our equation. x is no longer unknown, it's equal to 10.

But there's one more step in problem-solving. We have to take our answer and see if it meets the requirements of the original problem. We were told that in

5 years, Ted would be 15 years old. Does this work out right if Ted is 10 now? Yes, because $10 + 5=15$. So it checks.

This section

A. demonstrated how to solve the equation using the golden rule of equations, and check the answer,
or
B. demonstrated how to set up the equation given words that describe the problem?

321. Now let's do another problem involving the golden rule of equations. But for a preview: in this one, instead of subtracting, we're going to be dividing each side of the equation by the same number.

Here's the problem. When Jack is 5 times older than he is now, he will be 70. How old is Jack now?

Let's let x represent Jack's age now. If Jack were 10 now, then "5 times older than he is now" would be $5*10$ or 50. If Jack is x now, then "5 times older than he is now" would be 5x.

But we're also told that his age when he's 5 times older than he is now is 70. So we have two expressions for the same thing, and therefore they're equal to each other. So

$5x=70$ is our equation to solve.

We're looking for what x is. How do we get from 5x to x? We know that division is the inverse of multiplication, so let's try dividing by 5.

$(5x)/5$ is $(5x)* 1/5$. According to the associative and commutative laws of multiplication, that's also equal to
$x*(5*1/5)$ or $x*1$ or x. The 5's "cancel out" when we divide 5x by 5. So if we divide the left side by 5, that will give us x.

So let's follow the golden rule of equations and divide both sides by 5:

$(5x)/5= 70/5$
$x= 14$

So our answer is that Jack is 14 now. Let's check it with the requirements of the original problem. If Jack is 14 now, when he is 5 times older, he'll be 5*14 or 70. So it checks!

This section

A. gave an example of using the golden rule of equations by subtracting the same thing from both sides,
or
B. gave an example of using the golden rule of equations by dividing both sides by the same number?

322. If you're not sure how to represent something with literal numbers, it often helps to figure out how you'd represent that something with an ordinary numeral, any ordinary numeral you want. Here some examples:

How would we represent "Three more than x?" We might think, "Three more than 5 would be 5+3. So three more than x is x+3."

How would we represent "Seven less than x?" We might think, "Seven less than 10 would be 10-7. So seven less than x is x-7."

If x is a certain speed, in miles per hour, how would we represent 5 miles per hour faster? We might think, "5 miles per hour faster than 30 miles per hour is 30+5 miles per hour. So 5 miles per hour faster than x is x+5 miles per hour."

If x is a certain speed, how would we represent half that speed? How about (1/2)x or x/2.

How would we represent "twice as much as three less than x?" This is a tricky one. First, we figure that "three less than x" is x-3. Then to multiply that whole number by 2, we put parentheses around it: 2(x-3). To check our reasoning, we calculate "twice as much as three less than 10" and we satisfy ourselves that we first subtract 3 and then multiply by two.

How would we represent "three less than twice x?" Here's another tricky one. Twice x is 2x. Three less than that is 2x-3. To check our reasoning, we calculate three less than twice 8, and we satisfy ourselves that we first multiply by two and then subtract 3.

The main point of this section is that

A. twice as much as eight more than x is $2(x+8)$,
or
B. when you're not sure how to represent something with literal numbers, you can pick any ordinary non-literal number and represent it with that; that will give you a big clue about how to represent it with the literal number.

323. Sometimes we need several steps of doing the same thing to both sides of an equation in order to solve it. Let's look at a problem where there are two steps. We'll first subtract the same thing from both sides, and then divide both sides by the same thing.

Here's our equation to solve:

$$2x + 1 = 11$$

Let's first figure out what 2x is. Let's do that by subtracting 1 from both sides of the equation:

$$2x +1 -1 = 11-1$$

or $2x=10$.

Now that we've figured out what 2x is equal to, let's figure out what x is equal to. We can do that by dividing both sides by 2.

$$(2x)/2=10/2$$

$$x=5$$

So we get 5 as the value of our unknown. Let's check it by seeing if 5 works in our original equation, $2x+1=11$. When we substitute 5 for x, we get $2*5+1=11$, or $10+1=11$. It checks!

This section

A. defined the golden rule of equations,
or

B. gave an example of an equation that we solved by not one but two steps of doing the same thing to both sides?

Using the golden rule of equations to make different forms of formulas

324. We know that the area of a rectangle is equal to its length times its width. Using variables to express this gives us the familiar formula

A=LW,
or (since “equals” goes in both directions)
LW=A.

Now suppose we have the area of a rectangle, and the length, and we want to know the width. For example: the area of a rectangle is 10 square centimeters. The length is 5 centimeters. What’s the width?

We can substitute what we know into the formula
LW=A. L is 5 and A is 10, so

5W=10.

Then we can use the golden rule of equations to figure out what W is. We divide both sides by 5.

5W/5=10/5 or

W=2.

So we had a formula for finding area given length and width. But we used it to find width, given length and area. So we see that A=LW is not just for finding areas. With the golden rule of equations, we can find any one of the three things mentioned in the formula, if we know the other two!

This section gave an example of

A. using the golden rule of equations, and a formula, to find something other than what the formula seems to be written for,

or
B. the fact that you can subtract the same thing from both sides of an equation and the two sides will still be equal?

325. Let's suppose that we had lots and lots of problems where we were given the area of a rectangle and its length, and we needed to find its width. We might decide to use the golden rule of equations on our formula itself, to come up with a formula for the width! Here's how we would do it:

$LW=A$ is our formula for area, which is just writing $A=LW$ backwards.

Now let's divide both sides of our equation by L.

$LW/L= A/L$

On the left side, the L's cancel out, and we get

$W=A/L$.

So now we know that any time we have the area and the length of a rectangle, and we want to find the width, we just divide the area by the length! We have now a new formula, that's very similar to our original one of $LW=A$. We've just moved things around a bit to make it more convenient to find W knowing A and L. This is called "solving for W," or "finding W in terms of A and L."

This section gave an example of

A. solving an equation for one of the variables in a formula, in "terms of" the other variables,
or
B. finding the perimeter of a rectangle given its length and width?

326. Suppose we wanted a formula for the length of a rectangle when we know its area and its width. We start with

length * width = area

and divide both sides by the width. We get

length * width /width = area /width
length = area /width
or
L = A/ W

The formula this section talked about lets you find the

A. area when you have the length and width,
or
B. length when you have the area and width?

327. There are many, many examples of formulas that relate three variables, where one of them is the product of the other two. For all these formulas, you can find any one of the three numbers when you are given the other two, by using the golden rule of equations. Here's another example. Suppose you go at an average rate of 3 miles an hour, for 2 hours. How far have you gone? The answer is 3 x 2, or 6 miles. This is an example of the principle that rate x time =distance. We can express this as a formula by saying

$rt=d$.

Now, suppose we are given the distance and the time, and we want to know the rate? For example, we go 10 miles in 2 hours. How fast have we gone? We can solve our rt=d formula for rate. We do this by dividing both sides by time, or t. When we do this, we get

$rt/t=d/t$

and when we cancel out the t's, we get

$r=d/t$.

Now we have a new formula that tells us that whenever we have a problem where we are given a distance someone went and how long it took them, we can find the average rate by dividing the distance by the time. We “solved” the original formula for r, or we “put r in terms of d and t.”

We can use our new formula to answer the question. If we’ve gone 10 miles in 2 hours, our rate is 10/2 or 5 miles per hour.

This section

A. gave another example of using a formula, plus the golden rule of equations, to find one of the three variables given the other two,
or
B. explained why canceling works?

328. There’s one more formula that comes from the rt=d formula. If we divide both sides by r, we get t=d/r, or

time = distance /rate.

Here’s an example of how to use that formula. Someone goes 15 miles. His average speed is 3 miles an hour. How long does it take him? We divide the 15 miles by the 3 miles per hour, to get that it takes him 5 hours. To check, we can multiply rate by time and see if we get the right distance. 3 miles an hour times 5 hours does give us 15 miles, so it does check.

This section solved a problem that used the formula

A. rate= distance/ time,
or
B. time = distance/rate?

Using the golden rule of equations to turn repeating decimals into fractions

329. Now that we know how to use variables, and how to use the golden rule of equations to solve equations, we can do a very neat trick: we can change repeating decimals into fractions. Let’s consider the repeating decimal .3333....
Let’s call that number x; x=0.33333.... Now, what’s 10x? To multiply any number by 10, we move the decimal point one place to the right. So 10x=3.33333....

Now let's write those two equations:
$10x=3.3333....$
$x= 0.3333....$
Our golden rule of equations says that we can subtract the same thing from each side of an equation and get a true equation. Now if $x=0.3333....$, x and 0.3333... are the "same thing." So let's subtract x from $10x$, and 0.3333... from 3.3333.... We get

$9x=3$

because when we took $10x-x$, we got $9x$, and when we took 3.333... -.3333..., we got 3.

Now we can solve this equation by dividing both sides by 9:

$x=3/9$ and reduce this to $x=1/3$.

So we've found that .33333... is the same thing as 1/3!

You can go through this process with any repeating decimal!

The point of this section was to

A. illustrate a strategy to change repeating decimals into ordinary fractions, or
B. illustrate a strategy for reducing fractions?

Chapter 16: Percents

Percent means hundredths

330. Percent, or %, means hundredths. If I say 50%, I mean 50 hundredths. If I say 1 percent, I mean one hundredth. If I say 75 percent, I mean 75/100 or 0.75. If I say 100%, I mean 100 hundredths or all of something or the whole thing or 1 thing.

This section

A. defined percent,
or
B. gave examples of various percent problems that come up in life?

What good are percents?

331. Why do we even need percents? We can say anything that we can say with percents just as accurately with decimal fractions or ordinary fractions. Instead of advertising a 25% discount, a store could advertise a .25 discount or a 1/4 discount. Instead of saying that a bank pays 5% interest, one could say that it pays .05 interest or 5/100 or 1/20 interest.

So why even have this chapter in our book? There are two reasons. First, people are already thoroughly in the habit of communicating in terms of percents, so we'd better get used to that custom. Everything from television advertisements to newspapers to scientific articles talks in terms of percents.

But the second reason for this chapter is that speaking in terms of percents is, in my opinion, a good custom! Why? It's good because it lets us compare fractions more easily, with a "common denominator" of 100. For example: if I'm looking for a loan, and one bank will charge me 50 thousandths interest, another 1/21 interest, and another "point zero four eight" interest, I have some converting to do in my mind, or on paper, before I realize which one is the best deal. On the other hand, if the first advertises 5%, the second 4.76%, and the third 4.8%, I can compare them without doing any arithmetic!

One of the points in this section is

A. that there are two ways to figure out a new price given an old one and a "percent discount,"
or
B. that the author thinks percents are good things because they make fractions easier to compare?

Changing decimal fractions to percents

332. If percent means hundredths, then .18, which is eighteen hundredths, is equal to 18%. 0.01, which is one hundredth, is equal to 1 percent. What did we do to the decimal points in the decimal fractions .18 and .01, to change them to percents? We moved the decimal points 2 places to the right. And that's what we always do when changing decimals to percents: the decimal point goes two places to the right and a percent sign (or the word, *percent*) gets attached!

The main point of this section was

A. percents make fractions like .18 and 1/100 easier to compare,
or
B. to change any decimal fraction to a percent, move the decimal two places to the right and attach a percent sign?

Numbers over 1 get changed into percents over 100

333. The rule about moving the decimal point two places to the right doesn't change if we start with a number like 1.5 or 2.7. If the price of something went up by 1.5 times the original price, then it went up by 150% of the original price. If someone's speed on math facts is 2.7 times what they started out, then their current speed is 270% of their original speed.

There are certain situations where percents over 100 don't make sense. For example, "Of the students in Mrs. Grundy's math class, 150% of them failed the test." That doesn't make sense because you can't have more of them fail the test than are in the class in the first place! On the other hand, "They marked up the price by 150%" makes perfect sense, because you can increase the price of something by more than the thing originally cost, such as when you buy something for $10 and sell it for $25, increasing the price by $15.

This section

A. explained that when decimal numbers are over one, you use the same rule of moving the decimal point two places to the right, and you get percents over 100%,
or
B. explained how you change percents back into decimal numbers by moving the decimal point two places to the left?

Changing from percents to decimals

334. What's the "inverse function" of changing from a decimal to a percent? It's changing from a percent to a decimal. The inverse function gets you back where you started. What's the inverse function of moving the decimal point two places to the right? Moving it two places back to the left. That gets the decimal point back where it started.

So to change 18% to a decimal, the decimal point goes two places to the left, and it becomes .18, or eighteen hundredths. To change 1% to a decimal, we move the decimal point two places to the left, and it becomes .01. (Of course, if a whole number doesn't have a decimal point, we put one in or imagine it just after the whole number. And if we're moving the decimal point more places than there were originally in the number, we add zeroes as place holders.)

To change any percent to a decimal number, just move the decimal two places to the left, and get rid of the percent sign.

A consequence of the rule stated in this section is that

A. 0.04=4/100
or
B. 78%=0.78?

Adding and subtracting percents

335. This is an easy one, because there's nothing new to learn – you just add and subtract the percents as you would any other numbers, and your answer is in percent form also. For example: The interest a bank pays is 5%, and over the

course of the year, the rate goes up 2%. How much is it at the end of the year? 7%. The next year it goes down 3%; after that it's 7% - 3% or 4%.

The point of this section is that

A. you don't need to do anything special to add and subtract percents,
or
B. it's important to read carefully the language of the per cent statement you're dealing with, so that you can answer the question, "percent of what?"

Of means times for percents as well as fractions

336. What is 30% of 200? This means the same thing as, "What is 30 hundredths times 200?" A percent of a number is the same as that percent times the number.

This section implies that

A. just as 1/2 of 10 is 1/2 * 10, 50% of 10 is 50% * 10,
or
B. it's useful to remember that part/whole = percent?

When you multiply or divide a percent, change to a decimal or fraction first

337. If we want to know what 50% of 10 is, we multiply, because of means times. But first, we change 50% to a decimal or a fraction. 50% is .50 or .5; it's also 1/2. If we multiply any of these by 10, we get our answer, which is 5!

The main point of this section is
A. of means times,
or
B. before multiplying or dividing by a percent, change the percent to a decimal or fraction?

Three types of percent problems

338. The first type of percent problem just uses the idea that “of means times.” What is 25% of 32? To find this, we just multiply 25%, or .25, times 32. Sometimes this type of problem is put in more words. For example, “75% of the people in a certain college class who applied to medical school were accepted. If 36 applied, how many were accepted?” Again, we just use “of means times” and “percent means hundredths,” and multiply .75 by 36 to find our answer.

This section was about a certain type of problem where we are asked to

A. find a certain percent of a number,
or
B. change a decimal to a percent?

339. This first type of problem may be called a problem where we use the principle that

part=whole * percent.

We are given the size of a whole set, and told that a certain percent did something, and we want to find the number that did that something. Here’s another example: There are 50 people living on our block. (That’s the whole.) 30% of them support candidate x in the upcoming election. (That’s the percent.) How many of them intend to vote for candidate x? (The part is what is asked for.)

If we use the formula part=whole * percent,

we get: part = 50 * .30, or our “part” is 15 people.

340. Now suppose we are given the part and the whole, and are asked for the percent? For example, 30 is what percent of 50? Of 50 people in a class, 30 of them passed a test; what percent of them passed? If

part = whole * percent,

let's "solve for" percent. First we'll switch left and right, to get percent on the left side:

whole * percent = part

And now let's divide both sides by whole:

whole * percent /whole = part/whole

The wholes cancel, so we get

percent = part/whole.

So to find, what percent is 30 of 50, we divide 30 by 50, getting .60, which is equal to 60%.

The main formula that was used in this section was

A. percent = part/whole,
or
B. whole = part/percent?

341. Here's the third type of percent problem. In this one, you are given the part and the percent, and you are asked to find the whole. For example: 20% of the people in a certain club showed up for a meeting. If 7 people were there, how many people are in the club?

Let's start with

percent * whole = part   and divide both sides by percent, to solve for whole.

We get

percent * whole /percent = part /percent,

or, after the percents cancel out on the left, whole = part /percent.

Now we have a third form of our formula that we can use when we are given a part and a percent and we need to find the whole.

Getting back to our sample problem: if 20% of our club consists of 7 people, then how many are in our whole club? 7 is the part, and 20% is the percent, which equals .20.

whole = part/percent
whole = 7/.20
whole = 35

The third type of problem

A. gives you the whole and the part and asks you for the percent,
or
B. gives you the part and percent and asks you for the whole?

342. Do you remember how we figured out three formulas relating the length, width, and area of a rectangle? They were:

length x width = area
length=area/width, and
width = area/length.

We also figured out three formulas relating average rate, time, and distance. They were:

rate * time = distance
rate = distance/time
time = distance/rate

In the recent sections, we have figured out three formulas relating whole, percent, and part. They are:

percent * whole = part

percent = part/whole
whole = part/percent

Do you see how the thinking involved in these situations is almost exactly the same? When you study science, especially physics, you will run into lots more formulas like these. The more familiar you get with this general way of thinking, the better you'll be able to do lots and lots of different problems.

The main point of this section is that

A. percent * whole = part,
or
B. the relationships among percent, whole, and part, and among rate, time, and distance, and among length, width, and area, are all very similar, requiring very similar types of thinking?

343. Remember the multiplication and division fact families? Here's an example of one of them:

5*7=35
7*5=35
35/7=5
35/5=7

Please compare that to the following four formulas:

percent * whole = part
whole * percent = part
part/whole = percent
part/percent = whole

Or to the following four formulas:

rate * time = distance
time * rate = distance
distance/time = rate
distance/rate = time

I hope you see that there is a “family” of formulas, just as there are fact families.

The purpose of this section was to

A. show the similarity between multiplication and division fact families and different forms of formulas having to do with percent or rate,
or
B. to point out that rate * time = distance?

Taking a “percent off”

344. You will very often see situations in which a certain number is reduced by a certain percent. For example, a desk that costs $200 has its price reduced by 25%. What is the new price?

First let’s say, there are two correct ways of solving this problem. Here’s the first way. The first important thing to realize is what people mean when they say, “The price is reduced by 25%,” or “There is 25% off.” This is really a shortened way of saying, “The original price has subtracted from it, a discount of 25% *of the original price*.” The percent, in order to be meaningful, has to be a percent OF something, and that something is whatever we started out with!

So what is the amount that’s subtracted from the original price? It’s $200 x .25, or $50. This is the dollar amount of the discount, or the amount to be subtracted.

After we find that dollar amount, we just subtract it from the original price. $200 minus $50 gives $150, which is the discounted price and the answer to our problem!

The steps in finding the discounted price described in this section were to

A. subtract the percent discount from 100%, and multiply what you get by the original price,
or
B. multiply the original price by the percent discount, and then subtract what you get from the original price?

345. There's a second way of doing these discount problems. The original price is 100% of the original price. If we take 25% of it away, we get 75% of the original price left. So our question becomes, what is 75% of the original price? To find out, we multiply $200 x .75, to get $150. We get the same answer that we got doing the problem the other way.

The steps in finding the discounted price described in this section were to

A. subtract the percent discount from 100%, and multiply what you get by the original price,
or
B. multiply the original price by the percent discount, and then subtract what you get from the original price?

Increasing a number by a certain percent

346. We use exactly the same reasoning when we increase a price or any number by a certain percent. Again, the crucial thing to remember is that the percent increase represents a percent *of the original number*. So suppose I'm a store owner, and I buy a desk for $200. I mark the price up by 25%. What's the new price?

In the first way of doing this problem, I find the dollar amount of the markup by multiplying $200 by .25; I get $50. Then I add $50 to the original price to get $250, the new price.

The steps in finding the marked-up price described in this section were to

A. add the percent markup to 100%, and multiply what you get by the original price,
or
B. multiply the original price by the percent markup, and then add what you get to the original price?

347. Here's the second way of doing this problem. We figure that the original price is 100% of the original price. When 25% of the original price is added to it, we then have 125% of the original price. So the question becomes, what is 125%

of $200? To find this, we multiply $200 x 1.25, and get $250. Again, we get the same answer with both methods.

The steps in finding the marked-up price described in this section were to

A. add the percent markup to 100%, and multiply what you get by the original price,
or
B. multiply the original price by the percent markup, and then add what you get to the original price?

Interest

348. If you have some money that you don't want to spend right now, you can get paid for lending it to someone else. That payment is called interest. Suppose that the interest rate is 5% per year. Suppose you lend someone $20,000 for a year. How much would you get paid at the end of the year, counting getting your original loan back, plus the 5% interest you earned?

This is exactly the same sort of problem as the "markup" problem we just solved. Again, we can do it in two ways. In the first way, we first compute the interest. Again, it's crucial to realize that the interest is a percent OF the original loan amount. So $20,000 * .05 or $1000 gives the interest. That added to the original $20,000 amount means that we get back $21,000.

This section described a method in which you

A. add the percent interest to 100%, and multiply what you get by the original loan,
or
B. multiply the original loan by the percent interest, and then add what you get to the original loan amount?

349. Here's a second way to solve the same problem. At the end of the year, you will get back 100% of your original money, plus 5% more, so you will get back 105% of your original loan. So 105% of 20,000 or 1.05 x 20,000 = $21,000.

This section described a method in which you

A. add the percent interest to 100%, and multiply what you get by the original loan,
or
B. multiply the original loan by the percent interest, and then add what you get to the original loan amount?

Chapter 17: Measures

Adding and subtracting measured numbers require the same units

350. If someone were to ask you, “What is four hours, plus three miles per hour,” what would you say? You can’t add the four plus the three and get 7 of anything meaningful. The question just doesn’t make sense. You can’t add time plus speed and get a meaningful answer. Similarly, what is 4 feet minus 10 pounds? This question also can’t be answered. Whenever you are asked to add or subtract numbers that are a certain number of units of a measure, the numbers have to be the same types of measures. You can’t add distance to weight or speed to time and so forth.

This section said that

A. If you are going to add or subtract measures, the numbers have to represent measures of the same sort of quantity, such as distance or time or weight or speed, etc.
or
B. Measurements can be converted into different units of the same type of quantity?

351. Here’s a different type of problem. Someone works for 5 hours, and then the person works for 120 more minutes. How long has the person worked altogether? This is a “some and some more” problem, that calls for addition. Can we add 5 hours and 120 minutes? We can, if we change one of the numbers so that both numbers are in the same units. We know that 60 minutes are in one hour, so 120 minutes make 2 hours. So instead of adding 5 hours + 120 minutes, we add 5 hours + 2 hours. We come out with 7 hours.

We could convert minutes to hours because they are both units of time. If we had been asked to add 5 hours plus 120 feet, we could never have converted feet to hours, because feet and hours are units of different types of quantities.

Which of the statements below has more to do with the points made in this section?

A. Feet, inches, and meters are units of the “same type of quantity,” namely length, whereas feet and seconds are units of different types of quantities,
or
B. Often when we read a ruler, we read a mixed number such as 8 3/4 inches or a decimal mixed number such as 4.7 centimeters?

You can often multiply and divide numbers that aren’t in the same units

352. Sometimes trying to multiply or divide numbers measured in different units doesn’t make much sense. Five pounds of dirt times 40 volts of electricity doesn’t give an answer that means much. However, at other times, multiplying and dividing numbers of different units makes lots of sense. For example, someone goes 10 miles in 2 hours. How fast does the person go? We can divide 10 miles by 2 hours and get 5 miles/hour. Notice that we divided 10 by 2 and got 5; we also divided miles by hours and got a new unit, miles/hour.

Suppose you have 25 pounds of dirt, and you lift it all 3 feet higher than it was. How much work have you done? You may learn when you study physics that the work you do in this situation is equal to the weight you lift, times the height that you lift it. So you do 3 feet x 25 pounds of work, or 75 foot-pounds of work! (You pronounce foot-pounds “foot pounds.”) You multiply two units, and get a different unit. A “foot-pound” is a unit of work!

This section makes the point that

A It doesn’t make sense to add and subtract numbers of different units,
or
B. It often makes lots of sense to multiply and divide numbers that are in different units.

353. Even when we are multiplying and dividing, we usually want the numbers that measure the same type of quantity to be in the same units. If we’re dealing with time in a certain problem, for example, we don’t want to leave some of the numbers in hours and others in minutes. For example:

Jim walks 3 miles an hour. How far does he walk in 120 minutes?

Hours and minutes are both units of time, so we should change one of them so they are in the same units. 120 minutes = 2 hours, so now we can change our problem to an easier one:

Jim walks 3 miles an hour. How far does he walk in 2 hours?

Now we multiply 3 $\frac{\text{miles}}{\text{hour}}$ * 2 hours = 6 miles.

Notice that we can "cancel" the hour units in the denominator and the numerator, leaving us with miles, just as we could cancel in a problem like

3/4 * 4 = 3.

One point this section made was that

A. when we are dealing with a multiplication or division problem that talks about two different measures of the same type of quantity (such as hours and minutes, both ways of measuring time), we usually want to change one of them so the units are the same,
or
B. changing from one unit to another makes us decide whether to multiply or divide?

Changing from one unit to another

354. You will often get problems where you have to change from one unit to another. For example:

(There are 3 feet in a yard.) 6 yards is how many feet?

In this problem, 6 yards is what we're wanting to convert to feet. 3 feet per yard is the "conversion factor." When converting units, you have to decide whether to multiply by the conversion factor, or divide by it. Do we divide 6 by 3 and get 2 feet, or multiply 6 by 3 and get 18 feet?

Because changing units is so important in mathematics, I want to tell you about three different ways to decide whether to multiply or divide by a conversion factor. Any one of them will give you the right answer. But I think your grasp of the subject is increased when you are familiar with all three of them.

The question this section raises is,

A. which conversion factors (like 16 ounces per pound, 100 centimeters per meter, etc.) should you memorize, and which should you look up?
or
B. how do you decide whether to multiply or divide by a conversion factor when you change units?

Method 1: Thinking about the larger unit as a set containing a certain number of the smaller unit

355. How do we ordinarily figure out whether to multiply or divide in word problems? We usually try to visualize what we're given as a certain number of sets, with a certain number in each set, making a total number. If we're given the total, we divide, and if we're given the number of sets and the number in each set, we multiply. To refresh our memory with an example:

There are 6 boxes of peanuts, with 3 peanuts in each box. How many peanuts are there in all?

Here we're given the number of sets (6 boxes) and the number in each set (3 peanuts), so we multiply to get the total number of peanuts. For a different problem:

There are 18 peanuts total. We want to divide them into groups with 3 in each group. How many groups do we make?

Here we're given the total and we want to find the number of sets, so we divide 18 by 3 to get 6 groups.

This section

A. explained a method of cancelling when you convert units,
or
B. refreshed our memory about thinking about whether to multiply or divide, depending on what we are given in a problem having to do with number of sets, number in each set, and total number?

356. We can decide whether to multiply or divide when converting units, using just the same thinking. Here's our problem:

There are 3 feet in each yard. How many feet are in 6 yards?

We can visualize this where each yard is a set containing 3 feet. There are 6 such sets. So we multiply the number of sets times the number in each set, to get the number of feet in all.

This section gave an example where

A. we multiplied the number of sets by the number in each set, to get the total,
or
B. we divided the total by the number in each set, to get the number of sets?

357. Now let's consider a different problem.

There are 12 feet total. Each yard is a set of 3 feet. How many yards are there?

The way this problem is expressed, it's easier to see that you're given the total and the number in each set; you divide 12 by 3 to get the number of yards. Suppose the problem had been expressed like this:

How many yards does 12 feet equal?

Then you might have to do some translating in your mind, to make it clearer to yourself that 12 feet is the total and there are 3 feet in each set making a yard, and you are being asked how many such sets there are.

In the example in this section,

A. we multiplied the number of sets by the number in each set, to get the total,
or
B. we divided the total number by the number in each set, to get the number of sets?

Method 2: Thinking, "If I convert to bigger units, there will be fewer of them," and vice versa

358. Let's imagine a certain distance that doesn't change: say the height of a certain book about 1 foot tall. If we measure the distance in smaller units, such as inches, there will be more of them, in this example 12 inches. If we measure in still smaller units, such as centimeters, there will be still more: about 30.5 centimeters. And if we measure in still smaller units, such as millimeters, there will be still more of them: about 305 millimeters.

On the other hand, if we measure the height of the book in bigger units, there will be fewer of them. If we measure in meters, there will be .305 meters. If we measure in kilometers, we find that the book is only a very small number of kilometers, 0.000305.

Does it make sense to you that if we're talking about the same distance, we would need more small units to make that distance, and fewer big units to make that distance?

Here's another example. A gram is about the weight of a paper clip, and a kilogram is about the weight of a fairly thick textbook. If we were to measure your weight in grams, we'd need many more of them to express the same weight than if we used kilograms, wouldn't we? In fact, we'd need a thousand times as many of the small units as we would need of the big units in this example, to tell your weight.

A major idea this section is trying to get across is that

A. if you're measuring the same quantity by big or small units, you would need more of the small units and fewer of the big units,
or
B. there are 1000 grams in a kilogram?

359. So if you're measuring the same distance in feet and yards, do you come out with a bigger number for feet, or yards? It would be bigger for feet, because a

yard is bigger than a foot, and you need more feet to represent the same distance. If you're measuring the same weight in grams and kilograms, which number comes out bigger? The number of grams, because it's a smaller unit, and you need more of them to represent the same weight.

Here's the rule: the bigger the unit, the smaller the number of them. The smaller the unit, the more of them.

Here's a different way to express that rule. When we change from a bigger unit to a smaller unit, the number of units gets bigger. For example: 2 yards equals 6 feet. The number of feet is bigger than the number of yards. When we change from a smaller unit to a bigger unit, the number of units gets smaller. For example, 9 feet equals 3 yards. Going to a bigger unit meant going to a smaller number of units.

A summary of this section is that

A. if you're changing from a big unit to a small unit, your number gets bigger, and vice versa,
or
B. you multiply or divide by conversion factors, and hardly ever do you do any adding or subtracting when you change units?

360. Here's an example of how we would use this type of thinking in deciding whether to multiply or divide by a conversion factor. Here's the problem.

There are two pints in a quart. How many pints are in 8 quarts?

So we think: we're starting with 8 quarts. We're wanting to change that to pints. 2 is our conversion factor. Do we multiply by 2, or divide by 2? A pint is less than a quart. So we will have more when we measure in pints than we did when we measured in quarts. So to get a bigger number, we multiply by 2 rather than divide by 2. 8 x 2 = 16 pints.

This section gave an example of a situation where

A. we changed from a bigger unit to a smaller unit, so we made the answer come out bigger?
or

B. we changed from a smaller unit to a bigger unit, so we made the answer come out smaller?

361. Here's another example.

There are 4 quarts in 1 gallon. 20 quarts= _____ gallons?

This way of writing the question makes clear that we are changing from quarts to gallons, and 4 is our conversion factor. We're changing from a smaller unit to a bigger one, so the number of units is going to get smaller. So to get a smaller number we should divide by 4. 20/4= 5 gallons.

This section gave an example of a situation where

A. we changed from a bigger unit to a smaller unit, so we made the answer come out bigger,
or
B. we changed from a smaller unit to a bigger unit, so we made the answer come out smaller?

Method 3: Multiplying or dividing so as to make units "cancel out" right

362. When you study physics and chemistry, you do lots of changing of units. Let's study the method of changing units that physicists and chemists usually use. First, let's think about conversion factors in a new way. When you have a equation like

1 meter=100 centimeters,

what happens when you divide both sides of this equation by 1 meter? You get

1 meter/1 meter = 100 centimeters/ 1meter.

Now the left side of this equation is something divided by itself. So 100 centimeters/1 meter equals 1!

Using the same reasoning, 1 meter/ 100 centimeters equals 1. 1 foot/12 inches equals 1. 12 inches/1 foot equals 1!

So how can we take something like 3 feet and multiply it by 12 inches/1 foot? Because we are really multiplying it by 1. Conversion factors generally equal 1.

The main point made by this section is that

A. you should memorize conversion factors like 16 ounces/pound and 1 kilometer/1000 meters,
or
B. conversion factors are really equal to 1?

363. When we use our third way of converting units, we multiply by a conversion factor that makes the units cancel correctly. Here's an example:

4 meters = _____ centimeters?

We know one conversion factor that is 1 meter/100 centimeters, and another that is 100 centimeters/1 meter. Which one do we pick? If we pick the second one, the meters in the numerator cancel with the meters in the denominator of the conversion factor, and we are left with centimeters. So that's the one we pick! Here's what it looks like:

$$4 \text{ meters} * \frac{100 \text{ centimeters}}{\text{meter}} = 400 \text{ centimeters}$$

Do you see how the meters are in the numerator and denominator, and cancel? This conversion factor multiplies 4 by 100.

On the other hand, if we had tried

$$4 \text{ meters} * \frac{1 \text{ meter}}{100 \text{ centimeters}}$$

Then meters and centimeters wouldn't have canceled. This conversion factor would have divided 4 by 100, but it wouldn't have yielded an answer in the right units. So we pick the conversion factor that makes the cancellation possible.

If we're changing from meters to centimeters, we want the factor with meters in the denominator. As a general rule, we want whatever conversion factor that has whatever we're changing from, in the denominator.

The point of this section is that

A. to change from unit A to unit B, you multiply by a conversion factor that has unit A in the denominator so that the units will cancel correctly,
or
B. to change from a larger unit to a smaller unit, you pick a conversion factor that will give you a bigger number?

364. Now let's give a quick example of using all three thought patterns for a problem. How many inches are in 4 feet? Or: 4 feet = _____ inches.

Here's the first method. Each foot represents a set of 12 inches, and there are 4 such sets. So to find the total inches, 4 x 12 = 48.

Here's the second method. I'm changing from feet to inches, and inches is a smaller unit than feet. So in going to a smaller unit, I'll need a bigger number. So I multiply 4 by 12 rather than dividing 4 by 12.

Here's the third method. My conversion factor would be either 12 inches/1 foot, or 1 foot/12 inches. I'm starting with feet, so I want feet to be in the denominator of the conversion factor so feet will cancel and I'll get inches. I multiply

$4 \text{ feet} * \dfrac{12 \text{ inches}}{1 \text{ foot}}$; the feet cancel and I get the answer in inches.

This section gave

A. an example of converting from a bigger unit to a smaller unit, using all 3 methods,

or
B. an example of converting from a smaller unit to a bigger unit, using all 3 methods?

365. Now let's look at another example with all three methods. How many minutes are in 180 seconds? Or 180 seconds = _____ minutes.

Method 1: Each minute is a set of 60 seconds. So I want to know how many sets of 60 seconds there are in 180 seconds. To find how many sets there are of one number in another, you divide. So 180 $\div$ 60 = 3 minutes.

Method 2: A minute is more time than a second. I'm changing from a smaller unit to a bigger unit, so I'll need fewer of them. Since my number will become smaller, I'll divide by the conversion factor of 60 rather than multiply by it. So 180 $\div$ 60 = 3 minutes. It makes sense that a certain number of seconds equals a smaller number of minutes.

Method 3: The conversion factor can be 1 minute/60 seconds, or 60 seconds/1 minute. Since I'm multiplying seconds (which are in the numerator) by my conversion factor, I want seconds to be in the denominator of my conversion factor so seconds will cancel. So I multiply

180 seconds * $\frac{1\,\text{minute}}{60\,\text{seconds}}$; the seconds cancel out and leave me minutes.
Multiplying by 1/60 is the same as dividing by 60.

This section gave

A. an example of using all three methods to change from a smaller unit to a bigger unit,
or
B. an example of using all three methods to change from a bigger unit to a smaller unit?

366. Which conversion factors should you memorize? The ones for the metric system are certainly easiest, because the conversion factors are powers of ten, with few exceptions. Plus, the prefixes tell you what the powers of ten are! Micro

means millionth; milli means thousandth; centi means hundredth, and deci means tenth. Kilo means thousand, mega means million, giga means billion, and tera means trillion. So

1 millimeter = .001 meter, or
1000 millimeters = 1 meter.

1 centimeter = .01 meter,
or 100 centimeters = 1 meter.

1 kilometer = 1000 meters.

These are the most common units of length or distance. For mass,

1 milligram = .001 gram, or
1000 milligrams = 1 gram.

1 kilogram = 1000 grams.

And for volume,

1 milliliter = .001 liter, or
1000 milliliters= 1 liter

1 kiloliter = 1000 liters.

This section gave

A. examples of how milli means thousandth and kilo means thousand,
or
B. the relationship between pounds and ounces?

367. What do you call the non-metric system? I prefer to call it the “old fashioned” system. Sooner or later people will probably abandon it for the less cumbersome metric system. People have done just this in almost all countries other than the United States. But it’s still necessary to learn a bunch of the old-fashioned conversion factors.

For length or distance:

12 inches = 1 foot
3 feet = 1 yard
36 inches = 1 yard
5,280 feet = 1 mile

For weight:

16 ounces = 1 pound
2000 pounds = 1 ton

For volume:

8 fluid ounces = 1 cup
2 cups = 1 pint
2 pints = 1 quart
4 quarts = 1 gallon

This section gave

A. conversions for the metric system
or
B. conversions for the old fashioned system?

368. Even in the metric system, not all the conversion factors are powers of ten. Where is the exception? It's in units of time. With clocks all over everywhere, people are just too entrenched in the old fashioned system to change to "kiloseconds" or "millihours," although fractions of a second are referred to, for example, as milliseconds for thousandths of a second. For units of time, we are all stuck with remembering that

60 seconds = 1 minute
60 minutes = 1 hour
24 hours = 1 day
365.25 days = 1 year

12 months = 1 year
7 days = 1 week
100 years = 1 century
52 weeks (plus a day or so) = 1 year

A point this section made is that

A. for units of time, a nanosecond means a billionth of a second,
or
B. for units of time, even the metric system uses conversion factors other than powers of ten?

Adding and subtracting "mixed measures"

369. Suppose there's a package that's 2 pounds and 14 ounces, and another that's 7 pounds and 9 ounces. How much do they weigh altogether? And how much more does the second weigh than the first?

Suppose that someone works for 2 hours 40 minutes on one day, and 5 hours 30 minutes the next day. How much has she worked altogether? And how much more time did she work the second day than the first?

These are the sorts of problems that ask you to add or subtract measurements expressed in mixed form, where there are some of one unit and some more of a smaller unit. In doing such problems, we use the same concepts of "regrouping" or "carrying and borrowing" that we used in adding and subtracting ordinary decimal numbers, and that we also used in adding and subtracting mixed numbers like 1 4/5 and 7 3/5.

An example of the point made by this section is that

A. Adding or subtracting quantities like 1 gallon 3 quarts and 5 gallons 2 quarts causes us to use regrouping as we did with ordinary numbers or mixed numbers,
or
B. it is possible to simplify certain problems with measures by expressing every quantity in only one unit?

370. How do we add 4 gallons 3 quarts and 5 gallons 2 quarts? One option would be to change everything to quarts. Since there are 4 quarts in a gallon, 4

gallons 3 quarts equals 4*4 or 16 quarts plus 3 more quarts, or 19 quarts. 5 gallons 2 quarts equals 5* 4 +2 or 22 quarts. Then we could add 19 and 22 and get 41 quarts. Then, to change back into gallons and quarts, we divide by 4. We get 10 gallons, and one quart left over. So our answer is 10 gallons 1 quart.

This section illustrated how to

A. use the metric system,
or
B. add mixed number measures by changing all measures to the same unit?

371. The method we just illustrated works fine, but it is a little cumbersome sometimes. Here's another method that can be easier. This time, to add 4 gallons 3 quarts to 5 gallons 2 quarts, we write the problem like this:

4 gallons 3 quarts
+ 5 gallons 2 quarts

and then we just add each unit separately:

4 gallons 3 quarts
+ 5 gallons 2 quarts
9 gallons 5 quarts

Now we notice that 5 quarts are more than 4 quarts, so we can make another gallon. We divide 4 into 5 to figure that 5 quarts is 1 gallon 1 quart. So we add 1 gallon 1 quart to the 9 gallons we already had, to get 10 gallons 1 quart. The computation is easier than the previous method, because we're dealing with smaller numbers.

The main point of this section is that

A. in adding mixed measures, we can just add each unit separately; if there are enough of the smaller unit to make more of the larger unit, we change that to a mixed measure and add to what we got of the larger unit,
or

B. in adding mixed measures, what we are doing corresponds to what we do when we add mixed numbers like 4 3/4 + 5 2/4?

372. While the memory of the problem above is fresh in our minds, let's recall how we added mixed numbers, like 4 3/4 + 5 2/4. We set them up like this:

$$\begin{array}{r} 4\ 3/4 \\ +\ \underline{5\ 2/4} \end{array}$$

And then we add the whole number parts and the fraction parts separately:

$$\begin{array}{r} 4\ 3/4 \\ +\ \underline{5\ 2/4} \\ 9\ 5/4 \end{array}$$

Now we notice that 5/4 is greater than 1, so we can make one more whole number out of this improper fraction. If we divide 4 into 5, we get 1 1/4. So we add 1 1/4 to 9, to get 10 1/4.

Can you see how our reasoning in adding mixed numbers is exactly the same as it was in adding mixed measures?

The point of this section was that

A. when we use the metric system, we usually just express the measure as one unit rather than as a mixed measure, for example 148 centimeters rather than 1 meter 48 centimeters,
or
B. our reasoning in adding mixed numbers is just the same as that when we add mixed measures?

373. Now let's look at how to subtract mixed measures. One person worked 5 hours 10 minutes, and another person worked 2 hours 50 minutes. How much longer did the first person work than the second? Again, if we wanted to, we could eliminate the mixed measure problem by changing everything to minutes. The first person worked 5*60 + 10 or 310 minutes. The second person worked 2*60 +50 or 170 minutes. If we subtract 310-170, we get that the first person

worked 140 minutes longer. To change 140 back into a mixed measure, we divide 140 minutes by 60 minutes, and get 2 hours 20 minutes.

This section

A. gave an example of subtracting mixed measures by converting both to the smaller unit first,
or
B. gave an example of subtracting mixed measures by subtracting each unit separately, regrouping if necessary?

374. A different way of subtracting mixed measures is by subtracting each unit separately. We set up the problem like this:

$$\begin{array}{rl}
 & 5\text{ hours }10\text{ minutes}\\
- & \underline{2\text{ hours }\ 50\text{ minutes}}
\end{array}$$

But when we start to subtract the minutes, we notice that 10 minutes is less than 50 minutes. So we unbundle one of the 5 hours into 60 minutes. That changes 5 hours 10 minutes into 4 hours and 70 minutes. So now we have a problem where we can subtract each unit separately, and we do so.

$$\begin{array}{rl}
 & \overset{4}{\cancel{5}}\text{ hours }\overset{70}{\cancel{10}}\text{ minutes}\\
- & \underline{2\text{ hours }\ 50\text{ minutes}}\\
 & 2\text{ hours }\ 20\text{ minutes}
\end{array}$$

This section gave an example of

A. “carrying” when adding mixed measures,
or
B. “borrowing” when subtracting mixed measures?

375. Again, let’s compare what we did to the very similar operation with mixed numbers that we discussed in a previous chapter. Suppose we are subtracting

  5 1/6
- 2 5/6

When we start to subtract the fractional parts, we notice that 1/6 is less than 5/6. So we unbundle one of the 5 ones, to make 6/6. This turns 5 1/6 into the equivalent number, 4 7/6. When we rewrite the problem, we can subtract separately the whole number parts and fractional parts, and we do so.

  4 7/6
  ~~5 1/6~~
- 2 5/6
  2 2/6

We get 2 2/6, which we would reduce to 2 1/3.

Is it easy to see that the thought process we went through in subtracting mixed numbers is just the same one we used in subtracting mixed measures?

The point of this section was to

A. illustrate a subtraction problem with mixed numbers, to show that the thought process is the same as in subtraction with mixed measures,
or
B. to illustrate a subtraction problem with mixed numbers for the first time in this book?

Figuring out how long between two times

376. Suppose someone started working at 10:40 in the morning, and finished at 6:10 in the evening. How long did the person work?

Like most mathematics problems, there are several ways of doing this. One way is by figuring out the time between several shorter intervals, and adding them up. You would think as follows:

10:40 to 11:00 is 20 minutes
11:00 to 12 noon is 1 hour
12 noon to 6 pm is 6 hours
6 pm to 6:10 is 10 minutes

So when we add up the hours and minutes separately, we get 7 hours and 30 minutes.

This method is very much like a method of subtracting. How much is 25 minus 9? You could think: 9 to 10 is 1. Up to 20 is 10 more. Up to 25 is 5 more. So 10 plus 5 plus 1 gives 16.

The method that was described for both subtraction or for finding the amount to time between two times

A. makes use of "regrouping:" or "borrowing,"
or
B. does not make use of "regrouping" or "borrowing?"

377. The second method involves subtracting mixed measures, as we discussed earlier. But first, we have to deal with the fact that our usual system of timekeeping has us start over again in the middle of the day. What time would 6:10 pm be if we didn't do that, but kept on going, after 12 noon, so that 1:00 pm would be 13:00, 2:00 pm would be 14:00, and so forth? This is the way time is kept in the military forces, and is referred to as military time.

So how do we change from standard time to military time? If 1:00 p.m. becomes 13:00, and 2:00 pm becomes 14:00, 3:00 pm becomes 15:00, it looks like the rule is that you add 12 hours to standard time in the afternoon, to get the military time.

With military time, any time of day means "the number of hours and minutes that have elapsed since midnight."

This section

A. spoke about the meaning of subtraction,
or
B. spoke about the way to change from standard time to military time?

378. Let's return to our problem: how much time is between 10:40 a.m. and 6:10 p.m.? Let's change 6:10 to military time; it becomes 18:10, and that means that 18 hours and 10 minutes have elapsed since midnight at that time. At 10:40 a.m., 10 hours and 40 minutes had elapsed since midnight. So to find out how much time elapsed between the two times, we subtract 10 hours and 40 minutes from 18 hours and 10 minutes. The problem looks like this:

$$\begin{array}{rr} & 18 \text{ hours } \ 10 \text{ minutes} \\ - & \underline{10 \text{ hours } \ 40 \text{ minutes}} \end{array}$$

We've already talked about how to "borrow" or "regroup" or "unbundle" an hour when there are not enough minutes to subtract the minutes conveniently. When we do this, we get the following:

$$\begin{array}{rr} & 17 \qquad\quad 70 \qquad\quad \\ & \cancel{18} \text{ hours } \ \cancel{10} \text{ minutes} \\ - & \underline{10 \text{ hours } \ 40 \text{ minutes}} \\ & 7 \text{ hours } \ 30 \text{ minutes} \end{array}$$

Which, reassuringly, is the same answer we got by the other method.

The method of telling how much time elapsed between two times described in this section involved

A. adding up the smaller intervals that made up the interval between the two times,
or
B. converting the afternoon time to military time, and then subtracting the earlier time from the later time, "unbundling" an hour into minutes if necessary?

Multiplying by mixed measures

379. Suppose we have a board 1 foot 3 inches long, and we lay 5 of such boards end to end. How long will the resulting string of boards be?

To solve this problem, we want to multiply 1 foot 3 inches by 5. How do we do this?

One simple way is to simplify things by changing the “mixed measure” of 1 foot 3 inches to a “pure measure” of inches. We would think, 1 foot is 12 inches, plus 3 more inches, equals 15 inches. Then we multiply 15 inches by 5, to get 75 inches. To change this back to a mixed measure, we divide by 12 inches per foot. When we divide 75 by 12 we get 6 remainder 3, or 6 feet with 3 inches left over. So our answer is 6 feet 3 inches.

If we’d wanted, we could have changed 1 foot 3 inches to a “pure measure” of feet: 1 3/12 feet, or 1 1/4 feet. Then we could have multiplied 1 1/4 feet by 5, coming out to 6 1/4 feet, or 6 feet 3 inches.

This section illustrated multiplying by a mixed measure, using the technique of

A. changing the mixed measure to only one type of unit, and then multiplying,
or
B. multiplying both numbers in the mixed measure by the multiplier, and then simplifying?

380. Another way to solve the same problem would be to multiply both the feet and the inches in the mixed measure by the multiplier, and then simplify. Here’s what this would look like.

$$\begin{array}{ll} & 1\text{ foot }\ 3\text{ inches} \\ \text{x} & \underline{\qquad\quad 5 \qquad\quad} \\ & 5\text{ feet }\ 15\text{ inches} \end{array}$$

Now we simplify by changing 15 inches to 1 foot 3 inches (by dividing by 12). We take that 1 foot 3 inches and add it to 5 feet, and get 6 feet 3 inches. So we get the same number that we got the other way.

This section illustrated multiplying by a mixed measure, using the technique of

A. changing the mixed measure to only one type of unit, and then multiplying,
or
B. multiplying both numbers in the mixed measure by the multiplier, and then simplifying?

Chapter 18: Ratios, Rates, and Proportions

381. When two numbers are different, there are two major ways of comparing them. One is by telling their difference, or how far apart they are, or what you get when you subtract one from the other. The other way of comparing them is by telling their *ratio*. The ratio of a first number to a second is how many times bigger the first is than the second, or what you get when you divide the first by the second.

Let's imagine that we are comparing 10 and 2. If we are concerned with their difference, we would say that 10 is 8 more than 2. If we are concerned with their ratio, however, we would say that 10 is 5 times more than 2, or that the ratio of 10 to 2 is 5. We get the ratio of 10 to 2 by dividing 10 by 2. Or, we make the fraction 10/2 and then reduce that fraction.

The point of this section was that

A. the ratio of x to y is how many times bigger x is than y, or x divided by y, or x/y,
or
B. when two ratios are equal, we call that equality a proportion?

382. There are several ways of telling ratios. We can use fractions, such as by saying the ratio of teachers to students is 1/8. We can use words, such as by saying that the ratio of teachers to students is 1 to 8. Or we can use a colon to mean the same thing, such as by saying that the ratio of teachers to students is 1:8. All these mean that there are 8 times as many students as teachers.

The purpose of this section was to

A. explain the meaning of a proportion,
or
B. tell three ways of communicating ratios?

383. If a certain school has a 3 to 1 ratio of girls to boys, that means there are 3 times as many girls as boys in the school. It also means that the number of girls, divided by the number of boys, comes out to 3. Ratios don't tell us how many of

each there are. The school with a 3 to 1 ratio of girls to boys could have 6 girls and 2 boys, or 30 girls and 10 boys, or 2700 girls and 900 boys, and so forth.

Suppose a different school has 20 girls and 60 boys. Now what's the ratio of girls to boys? If we divide 20 by 60, we get a number less than 1, a fraction, namely 20/60. When we reduce 20/60 we get 1/3. So this second school has a 1 to 3 ratio of girls to boys.

The purpose of this section was to

A. give examples of the types of proportions where when you increase one variable you decrease another,
or
B. give examples of ratios, and to make clearer how the word ratio is used?

384. Let's think some more about the school with 20 girls and 60 boys. We decided that the ratio of girls to boys was 1 to 3 or 1/3. What is the ratio of boys to girls? To get this, we divide the number of boys by the number of girls, and get 3. So the ratio of boys to girls is the reciprocal of the ratio of girls to boys.

It's always true that the ratio of a first number to a second is the reciprocal of the ratio of the second number to the first. Why is this true? Let's call the first number a, and the second number b. The ratio of the first to the second is defined as a/b. The ratio of the second to the first is defined as b/a. Since b/a is the reciprocal of a/b, the ratio of the second to the first is the reciprocal of the ratio of the first to the second.

An example of the main point of this section is that

A. if the ratio of girls to boys in a third school is 2 to 5 (or 2/5), the ratio of boys to girls in that school is 5 to 2 (or 5/2),
or
B. the ratio of girls to boys is a different number than the percent of students that are girls?

385. We've made the point already that if someone tells us a ratio of girls to boys, we can't give the exact numbers of girls and boys. However, we can use literal numbers to write expressions for the numbers of girls and boys. Let's say that the ratio of boys to girls is 2 to 3. This means that when we divide the number

of boys by the number of girls, we get a fraction that reduces to 2/3. This means that the actual number of boys and girls have to be numbers of the form 2x and 3x, because only fractions of the form 2x/3x reduce to 2/3. And because the number of boys and girls have to be whole numbers and not fractions of a person, x has to be a whole number. So in our school with a ratio of boys to girls of 2 to 3, there are 2x boys and 3x girls, where x is some whole number.

What's another example of the point made by this section?

A. When we double the speed of a trip, we cut the amount of time the trip takes in half,
or
B. if the ratio of dogs to cats in a certain animal shelter is 4 to 5, then we can say there are 4y dogs and 5y cats, where y is some whole number?

386. Suppose someone gives us a ratio and asks for a percent. For example, suppose we're told that the ratio of boys to girls is 2 to 3 at a certain school, and we're asked, "What percent of the students are boys?"

Here's how we'd do this problem. We know there are 2x boys and 3x girls, where x is some whole number. This means that the total number of students is 2x + 3x, or 5x. To compute the percent boys, we first divide the number of boys, not by the number of girls, but by the total number of students. So the fraction boys is 2x/5x. The x's cancel out, and we're left with a fraction of 2/5, or 40/100, which is the same as 40%. So a ratio of 2 boys to 3 girls means that 40% of the students are boys!

This section explained how to do a certain problem where you are

A. given the numbers of students, and asked for their ratio,
or
B. given the ratio of two sets of students, and asked for the percent who are members of one set?

387. We can go in the other direction with problems of this sort; that is, sometimes when we are given percents, we want to know ratios. For example: a certain shelter for dogs and cats has 30% dogs. What's the ratio of dogs to cats?

Let's call z the total number of dogs and cats at this shelter. The number of dogs then would be .30z. If 30% of the animals are dogs, then 70% of them are cats, and the number of cats is .70z. So the ratio of dogs to cats is .30z/.70z. When we divide top and bottom by z, and multiply top and bottom by 10, we get a ratio of 3/7. So the ratio of dogs to cats in this shelter is 3 to 7!

This section explained how to do a certain problem where you are

A. given a ratio and find a percent,
or
B. given a percent and find a ratio?

388. When two ratios are equal to each other we say that the quantities involved are directly proportional. The word proportion refers to the equality of the ratios. Here's an example. Let's think about the heights of objects such as trees or poles or sticks, and the heights of the shadows they make. Suppose you go outside at a certain time of day with a stick that is two meters tall, and you find that the shadow is one meter tall. Then you notice that the shadow of a stick one meter tall turns out to be a half a meter tall. Then you see that the shadow of a twelve inch ruler is six inches tall. From these observations, you infer that the ratio of the height of the stick to the length of the shadow at this time of day is always two to one no matter how long the stick is. You have inferred that the length of the shadow is directly proportional to the length of the stick.

A summary of this section is that:

A. if the ratio of one thing to another stays the same we say that the two quantities are directly proportional,
or
B. the distance you can travel in a certain time is directly proportional to the average speed at which you go?

389. When we know that two quantities are directly proportional, we can use that to find out things we didn't know. For example, suppose we want to know the height of a big tree, when we don't feel like climbing the tree with a long tape measure in our pocket. Suppose that at the same time of day in which we notice that a stick 2 meters long casts a shadow one meter long, we measure the length

of the shadow of this tree and find that it is 10 meters. Since the height of every stick is twice as long as its shadow, we can infer that the height of the tree is 20 meters. By using proportions, we found the height of the tree without doing any climbing!

In the example given in this section,

A. if you double the height of an object, you double the length of its shadow,
or
B. if you double the height of an object, you cut the length of the shadow in half?

390. In mathematics, we very frequently encounter problems like the one we just looked at, in which we know three of the quantities in a proportion, and we want to find the fourth. In the problem we just did, the three quantities we knew were the height of the stick, the length of its shadow, and the length of the tree's shadow, and we wanted to find the fourth quantity, the height of the tree. Let's think of another problem of the same sort. Now we are measuring at a different time of day, where the ratio of the shadows to the heights of objects is different. This time, a stick 2 meters tall casts a shadow that is 1.7 meters tall. The tree we're wanting to measure this time casts a shadow that is 13.6 meters tall. How tall is the tree this time? This time the problem is of exactly the same sort, but the numbers make the calculations a bit harder to keep straight. Before solving this problem, let's look at some ways people have made solving proportion problems easier for themselves.

The purpose of this section was to:

A. explain the meaning of inverse proportions,
or
B. set the stage for explanations to come of ways to keep things straight while solving proportion problems?

391. One of the things that is very useful to know when solving proportion problems is that you can "cross multiply." This means that when you have an equation where one fraction equals another, the numerator of the first times the denominator of the second equals the denominator of the first times the numerator of the second. In other words, suppose that we know that

$$\frac{a}{b}=\frac{c}{d}.$$

Cross multiplying tells us that

ad = bc.

How do we know this is true? All we have to do is to use the golden rule of equations and multiply both sides of the first equation by bd. On the left side, the b’s cancel out, and you get ad. On the right side, the d’s cancel, and you get bc.

Proving that you can cross multiply when you have an equation where two fractions are equal

A. involves complicated math that we haven’t discussed yet,
or
B. involves multiplying both sides of the equation by the same number?

392. Another thing to do that will help you keep proportion problems straight is to write the fraction in words before sticking the numbers in. So, in the example we gave above, we would first write

height of pole/ length of pole shadow = height of tree/ length of tree shadow.

Having written this, it is now easy to substitute the numbers in. If we call the height of the tree x and substitute in the other three quantities we are given, we get

2/1.7 = x/13.6.

If we cross multiply, we get 1.7x = 2*13.6. Then dividing both sides by 1.7, we get x = 2*13.6/1.7, or (using a calculator) 16. So the tree in this problem is 16 meters tall.

The purpose of this section was to

A. illustrate how to keep things straight in a proportion problem,
or
B. explain why cross multiplying actually makes a little extra work in this problem, as compared to simply multiplying both sides by 13.6?

393. Many problems involve what are called *rates*. Rates are simply ratios of certain sorts, where for example you are looking at how much of something happens in a given amount of time, or how much money is spent for a given unit of a good or service. For example, if someone can read ten pages of a book every four minutes, the rate of reading is 10/4 or 2.5 pages per minute. If a hiker goes twelve miles in four hours, his average rate is 12/4 or 3 miles per hour. If Mr. Jones gets paid a hundred dollars for working four hours, his average rate of pay was 100/4 or 25 dollars per hour.

The purpose of this section was to

A. explain why rate times time equals distance,
or
B. give some examples of ratios that we call rates?

394. You will encounter lots of problems that we might call "at the same rate" problems. For example: Mrs. Jones drives 200 miles in 4 hours. If she keeps going *at the same rate*, how long will it take her to go 500 miles?

There are two ways to think about such a problem. One is to simply treat it as a proportion problem. The distance she goes is directly proportional to the time she drives. So we can write the proportion

distance1/time1 = distance2/time2.

Then we substitute in our knowns and write x for our unknown, to get the following equation:

200/4 = 500/x.

We solve that equation for x, and we have our answer.

The second way to think about such a problem is to first calculate what the rate is. Since rate = distance/time, her rate is 200 miles/4 hours or 50 miles per hour. Now that we know this, we can say that since rate = distance/time, 50 miles per hour = 500 miles/x. We solve this equation to get our answer. The two ways of thinking about the problem are, as you can see, very similar.

The purpose of this section was to

A. explain why when you are given an "at the same rate" problem, you should always calculate the rate,
or
B. illustrate two ways of setting up an "at the same rate" problem?

395. When two quantities are directly proportional, one gets bigger as the other gets bigger. In fact, if you double one, you double the other; if you triple one, you triple the other, and so forth. However, some quantities behave in just the opposite manner. If you double one, you cut the other one in half. If you triple one, you divide the other one by three. When quantities are related in this way, they are said to be inversely proportional. Here's an example. Let's think about how long it takes to go 100 miles at various speeds. If we go 10 miles per hour, it will take us 10 hours. If we double the speed, and go 20 miles per hour, our trip will take us half as long: only 5 hours. If we go 50 miles per hour, our time will be even less: 2 hours. The higher the speed, the less time the trip takes, and the more time the trip takes, the lower is the speed. Speed and time are inversely proportional in this example.

The purpose of this section was to

A. define an inverse proportion and give an example of it,
or
B. show how long a trip of 100 miles takes at 20 miles per hour?

396. When two quantities are directly proportional, they have the same ratio. We can express this mathematically by saying that if x is directly proportional to y, then

$x/y = k$,

where k is a constant, or a number that stays the same.

On the other hand, when two quantities are inversely proportional, what stays the same is not their ratio, but their product. In our previous example, the product of the speed and the time always equalled the distance, which stayed constant at 100 miles. So for an inverse proportion: if x and y are inversely proportional, then

$xy=k$,

where k is a constant.

The purpose of this section was to

A. give another example of an inverse proportion,
or
B. contrast the formulas for direct and inverse proportions, which are $x/y=k$ and $xy=k$, respectively?

Chapter 19: Negative Numbers

397. What happens when you make numbers smaller and smaller, and get down to zero: can you keep going smaller? Can you make numbers that are less than zero? The answer is yes, and often these numbers have practical and understandable meanings. They are called negative numbers. The integer just below 0 is –1, called “negative one.” Below that are –2, -3, and so forth. Here’s what the number line looks like when we include some negative numbers:

-10 –9 –8 –7 –6 –5 –4 –3 –2 –1 0 1 2 3 4 5 6 7 8 9 10

This section made the point that

A. there are numbers less than zero, called negative numbers,
or
B. there are numbers that you can’t express with any fraction; these are called irrational numbers?

398. One way in which negative numbers make lots of sense is thinking about temperatures, using either of the temperature scales in most common use (Celsius or Farenheit). Suppose it is 4 degrees, and then it gets 7 degrees colder. What’s the temperature then? We can use the number line above to tell the answer. We can start at 4, and start counting off jumps to the left. When we have made 7 jumps to the left, we find we have wound up at –3. And that’s how cold it is, negative 3 degrees, or 3 degrees below zero. A way of stating the subtraction fact we just illustrated is that

4 - 7= -3.

So we find that not only are there numbers below zero. We can subtract bigger numbers from smaller numbers, and when we do so, we get one of those numbers below zero!

This section suggests that

A. you can’t take 10 from 5,

or
B. you can subtract 10 from 5, and sometimes the answer you get is very meaningful?

399. Now let's suppose it's 3 degrees below zero, and then it gets 5 more degrees colder. We would start at –3 degrees and make 5 jumps to the left. This would make us wind up at –8 degrees, which we call negative 8 degrees or 8 degrees below zero.

We would write the math fact we just illustrated as follows:

$-3 - 5 = -8$.

The situation this section talked about illustrates that

A. you can take away a number from a positive number smaller than it,
or
B. you can start with a negative number and take away from it, to make the answer even more negative?

400. Here's another way in which negative numbers make sense. Suppose you own only two dollars. Then you spend 6 dollars, by handing someone two dollar bills, and owing the person the rest of the money. How much money do you own now? If you start on the number line at 2, and make 6 jumps to the left, you wind up at –4. So your total money is negative 4 dollars. This means that not only do you not have any money, but even when you get some, you have to pay back the loan you've gotten for four dollars. You own less than 0 dollars.

This illustrates another common meaning of negative numbers. Positive numbers mean owning money for yourself, and negative numbers mean owing money to someone else.

This section

A. stated a general rule about combining numbers when one or more of them is negative,
or
B. gave another illustration of how negative numbers are meaningful in real life?

401. Here's another example of how negative numbers are used. Suppose you are going straight east, at 10 miles an hour. Suppose we define eastward speed as positive. Then your speed becomes more and more westward. Your speed gets lower and lower, and then reaches 0. Can your speed get even more westward than this? Yes, by actually moving to the west. So if you go 5 miles per hour west, you are going –5 miles per hour, using the system that was set up where east was positive. Negative distances and speeds and so forth are used all the time in physics. One direction is defined as positive, and the other direction is negative.

This section

A. explained how to multiply two negative numbers by each other,
or
B. gave another example of how negative numbers can be meaningful?

402. What do we call numbers like 7, 6, and 23, that aren't negative numbers (and aren't zero, either)? They're positive numbers. If we want to make very clear that they are positive numbers, we can write them with a plus sign: +7, +6, and +23 are positive 7, positive 6, and positive 23. +7 means exactly the same thing as 7.

The number 0 is neither positive nor negative. Zero is the point that separates all the positive numbers from all the negative numbers.

This section made the point that

A. we can call ordinary numbers like 8 and 9 positive numbers and write them as +8 and +9,
or
B. the sum of two positive numbers has to equal a positive number?

403. Every number other than zero has what's called its "opposite." The opposite of a positive number like 7 is –7. The opposite of a negative number like –20 is +20. To find the opposite of any number, you change the sign, and leave everything else the same. The opposite of x is –x, and the opposite of –x is x.

What's a consequence of what you were told in this section?

A. The square of any negative number is positive,
or
B. the opposite of any negative number is positive?

The meaning of the absolute value *of a number*

404. Suppose we've defined the direction east as positive. If you're driving east at 10 miles per hour, you're going +10 miles per hour; if you're going west at 10 miles per hour, you're going –10 miles per hour. Suppose we want to speak about "how fast you're going, ignoring which direction you're going?" (This is called your speed, whereas the number that has a sign depending on which direction you're going is called your velocity.) If we want to change velocity to speed, then we would ignore the negative signs of any negative numbers, or rather change those signs to positive. Changing to positive any signs of negative numbers, but leaving any positive numbers positive, is called taking the absolute value of a number.

The absolute value of +10 is +10, and the absolute value of –10 is also +10. The absolute value of 3 is 3, and the absolute value of –3 is 3. (3 and +3 are the same thing.)

So to take the absolute value of a number, you leave the value alone if it's positive, but change it to its opposite if it's negative.

A point made by this section is that

A. the absolute value of a positive number such as 25 is 25,
or
B. when you add a positive and a negative number, you subtract their absolute values?

405. The sign that we use for the absolute value is to enclose the number inside two vertical lines. So the absolute value of –7 is written |-7|. Here are some examples of true statements involving absolute values:

|-4|=4
|-10|= +10
|4|=+4
|10|=10

$|-3|=3$
$|-10|=+10$

The point of this section was to

A. define the meaning of the idea of absolute value,
or
B. state and illustrate the symbol for absolute value?

406. You will run into equations to solve that involve absolute value. Here's an example:

$|x|=10.$

What's the set of numbers, all of which satisfy that equation? The absolute value of 10 is 10, and the absolute value of –10 is 10 also. So the equation has two solutions: -10 and 10.

This section
A. gave an example of solving an inequality involving an absolute value,
or
B. gave an example of solving an equation involving an absolute value?

407. A few sections ago we said that in taking the opposite of a number, you change the sign, but leave "everything else" the same. Now that we have defined the phrase *absolute value*, we can say what the "everything else" is: it's the absolute value. To find the opposite of any number, you change the sign, but leave the absolute value the same. So if we change –5 to 5, we change the sign, but the absolute value remains 5. Also, if we change 5 to –5, we change the sign, but the absolute value of –5 remains 5.

What's the rule this section expresses?

A. To find the opposite of any number, you change the sign and leave the absolute value the same,
or

B. to add two negative numbers, you add their absolute values and make the answer negative?

Both subtracting and adding a negative number result in moving left on the number line

408. What do we mean by adding a negative number to another number? For example, what is 3 + (-7)? We define the meaning of our symbols so that both 3 – 7 and 3 + (-7) mean the same as starting at 3 on the number line, and taking 7 jumps to the left, or in the negative direction. Subtracting 7 and adding –7 have the same result. With both operations, you wind up at –4.

This section made the point that

A. x – y is the same as x + (-y),
or
B. x + y is the same as y + x?

409. We can look at every subtraction problem as an addition problem. 10 – 7 is just 10 plus negative 7. 4 – 6 is just 4 plus negative 6.

One really big advantage of starting to look at any subtraction as an addition of a negative number is that we can apply the commutative law to expressions like 10 – 7. Subtraction is not commutative, so 10-7 does NOT equal 7-10. But addition is commutative. So 10 + (-7) DOES equal –7 + 10.

To express this idea more generally, we can say that a – b is equal to –b + a, according to the commutative law of addition.

Why can every subtraction problem be regarded as an addition problem?

A. Because the associative law of multiplication says that grouping doesn't make a difference when you multiply three numbers together,
or
B. Because subtracting a number is the same as adding the opposite or negative of that number?

Subtracting a negative number results in a move to the right on the number line

410. What does it mean to subtract a negative number? In the previous section we said that subtracting a number is the same as adding the opposite of it. Subtracting 3 is the same as adding –3. If we want to keep things consistent, we'd better make subtracting –3 the same as adding its opposite, also. The opposite of a negative number is a positive number. So subtracting –3 is the same as adding 3! So if you see two minus signs in a row, you can turn one of them vertically and make a plus sign from them! For example, 6- (-3) is the same as 6+3.

What illustrates the rule that this section stated?

A. x - -b = x+b,
or
B. x*b=b*x?

411. We know that subtracting a positive number moves us to the left on the number line. For example, when we have 3-5, we start at 3 and move 5 jumps to the left, ending up at –2.

We also know that adding a negative number moves us to the left on the number line. For example, when we have 3 + (-5), we start at 3 and move 5 jumps to the left, ending up at –2. 3+ -5 is just the same as 3-5.

But subtracting a negative number moves us to the right on the number line. 3 – (-5) means that we start at 3 and move 5 jumps in the opposite direction we'd move for 3- (+5). And that direction is to the right. So we end up at 8. And this stands to reason, since 3 – (-5) is the same as 3+5!

This section

A. emphasizes that subtracting a negative number results in a move to the right on the number line,
or
B. defines what the opposite of a number is?

Rules for adding and subtracting numbers with positive or negative signs

412. Let’s figure out the rules for adding any combination of positive or negative numbers. From long ago we know how to add two positive numbers like 8 and 4 to get 12. But let’s say what we did in a way that also will apply to negative numbers. When we added 8 and 4, (two numbers that had the same sign) we added the absolute values of the numbers, and the answer had the same sign as the addends.

Another way of stating the rule given here is that

A. when you multiply two positive numbers, the result is always positive,
or
B. when you add two positive numbers, you add their absolute values, and make the sign of the sum positive just as the two addends are?

413. Now let’s see if the same rule holds true when adding two negative numbers. Do we add the absolute values of the two numbers, and give the answer the same sign as the addends? Let’s look at the number line:

-10 –9 –8 –7 –6 –5 –4 –3 –2 –1 0 1 2 3 4 5 6 7 8 9 10

Let’s add –1 and –3. We can start at –1 and make 3 jumps to the left. We end up at –4. That’s the same as adding the absolute value of –1 and –3 to get 4, and then appending the sign of the addends which is a negative sign. So the rule works.

How about if we had –3 and –4? We start at –3 and make 4 jumps to the left, ending up at –7. Again, we added the absolute values, and stuck a negative sign in front of the result because the addends were both negative.

The rule holds for every time when you are adding two negative numbers.

The rule of this section is that

A. a negative number times a negative number gives a positive number,
or

B. when you add two negative numbers, you add their absolute values, and give the sum the sign of the addends?

414. So whenever you add two numbers of the same sign, you add the absolute values, and give the sum the sign of the addends. It works for positive numbers (as in 8+4=12) and it works for negative numbers (as in –8 + -4 = -12).

The purpose of this section was to

A. give rules for adding numbers of unlike signs,
or
B. restate the rules of the previous two sections in a way that applies both to adding two positive numbers and to adding two negative numbers?

415. Now what about adding numbers of unlike signs? We remember a problem like 8 + (-5) is just the same as 8-5, and the answer is 3. We subtracted the absolute values, and the answer got the sign of the number with the higher absolute value. 8 was the higher one, so the difference got a positive value.

The rule this section illustrated is that

A. when adding numbers of opposite sign, subtract the absolute values, and give the difference the sign of the number with larger absolute value,
or
B. when adding numbers of like sign, add the absolute values, and give the sum the sign of the addends?

416. Now let's see if the rule we stated in the last section also holds up when the negative number is larger than the positive number. How about 3 + (-4)? Let's start at 3 on the number line, and move four jumps to the left. Or, we could start at –4 on the number line, and make 3 jumps to the right. In either case, our answer comes out to be –1.

____
-10 –9 –8 –7 –6 –5 –4 –3 –2 –1 0 1 2 3 4 5 6 7 8 9 10

So the answer is the difference between the absolute values of the two numbers, and the answer took on the sign of the number with the larger absolute value, which was –4.

Let's try it with another pair of numbers of opposite sign. How about 3 +(-5)? We could start at 3 and make 5 jumps to the left, ending up at –2. Or we could start at –5 and make 3 jumps to the right, ending up again at –2. Let's see if our rule worked out. 2 is the difference in the absolute values of –5 and 3, and it has a negative sign because –5 has a greater absolute value than 3. So it works again. It will work every time.

The purpose of this section was to

A. illustrate the rule that when adding numbers of unlike signs, you subtract the absolute values and give the answer the sign of the addend with the higher absolute value,
or
B. illustrate the rule that when adding numbers of like signs, you add the absolute values and give the answer the sign of the two addends?

417. So with these two rules, plus the fact that subtracting a number is the same as adding its opposite, we can subtract, as well as add, any possible combinations of positive and negative numbers.

What about a subtraction problem like 3-8? We can think of this as 3 + (-8), and just apply the second rule: we subtract the absolute values and give the answer the sign that went with –8. So we get –5.

What about a subtraction problem like –3-6? We can think of this as –3 + -6, and apply the first rule. We add the absolute values and use a negative sign because this is the sign of both addends. So we get –9.

What about a subtraction problem like 3 – (-6)? We remember that subtracting a number is the same as adding its opposite, so we can change this into 3 + 6, or 9.

How about a subtraction problem like –2- (-5)? We take those two negative signs in a row and make a plus sign out of them, to change our problem to –2+5. Then we subtract the absolute values and give the answer the sign that +5 had, so we come up with +3.

The point of this section was that

A. We don’t need another set of rules for subtracting, because we can use the fact that subtracting a number is the same as adding its opposite, and use the rules for adding that we already have,
or
B. multiplying a positive number by a positive number gives a positive number, just as we have done many times before taking up the study of negative numbers?

Rules for multiplying and dividing numbers with positive and negative signs

418. There are 4 possibilities we can run into when multiplying positive or negative numbers:

positive times positive
positive times negative
negative times positive
negative times negative.

“Positive times positive” is of course the regular multiplication people learn first, such as 8 x 4 = 32 and 5*7=35. The answer is positive.

A summary of this section is that

A. knowing the multiplication tables is important for multiplying negative numbers,
or
B. we’re going to figure out rules for multiplication, and of the four possible combinations, we’ve already been using one of them: positive times positive gives positive.

419. Let’s figure out what a positive number times a negative number would be. Let’s remember that multiplication is repeated addition. So a problem like

3 * (-2)

is the same as -2 + -2 + -2.

Using our addition rule for adding numbers of like sign, we add these to get –6. So a positive number times a negative number yielded a negative number. The product is the product of the absolute values, with a negative sign in front. This works out no matter what the two numbers are, as long as one is positive and the other negative.

What about a negative times a positive, such as (-2)*3? We don't have to do anything new for this! By the commutative law of multiplication, this is the same as 3 * (-2), and we've already done this!

The rule this section illustrated is that

A. a positive times a negative gives a negative number,
or
B. a negative times a negative gives a positive?

420. Now let's think about what we get when we multiply a negative times a negative. First, though, let's return to the problem (-2) * 3. What happened to 3 when it was multiplied by a negative number? The sign of the answer was switched from that of 3. Multiplying 3 by negative two gave an answer with a sign OPPOSITE to that of 3. Multiplying 3 by any negative number would give an answer with sign opposite to 3.

Suppose we want to keep consistent in this rule that multiplying a number by a negative number yields a product with sign opposite from the number we're multiplying by a negative. Then if we multiply (-2) * (-3), we would have to make the product come out with the sign opposite to that of –3. So the only possible answer is +6. Our rule is: a negative times a negative gives a positive.

What's a consequence of the rule that was the main point of this section?

A. –4 * 3 = -12
or
B. –4 * (-3)= +12?

421. So our rules for multiplying positive or negative numbers come down to something very simple:

like signs: positive product
unlike signs: negative product.

To illustrate:

Like signs:

8*4=32
(-8)(-4)=32

Unlike signs:
(-8)(4)=-32
(8)(-4)=-32

The main point of this section is that when multiplying two numbers,

A. to find the product, you always have to multiply the absolute values of the numbers,
or
B. like signs yield a positive product and unlike signs yield a negative product?

422. Let's use the rules we have to think about the sign of the reciprocal of a positive or negative number. We remind ourselves that the reciprocal of any number is that which, when multiplied by the number, gives 1. So the reciprocal of a positive number like 6 is a positive number, namely 1/6, because 6 * 1/6 gives 1.

What about the reciprocal of a negative number like –6? The reciprocal, when multiplied by a negative number (namely -6) has to yield a positive number. What's the only type of number which, when multiplied by a negative, gives a positive? A negative! So the reciprocal of –6 has to also be negative. Let's check –1/6 and see if it works: -6 * -1/6 =1. It does work, because the product is 1.

And it works for any other number: the reciprocal of any positive number is positive, and the reciprocal of any negative number is negative.

A summary of the main point of this section is that

A. the reciprocal of any number can be formed by putting 1 as the numerator and the number as the denominator,
or
B. the reciprocal of any number has the same sign as the number?

423. What about division of positive and negative numbers? We use two facts to keep us from having to do much additional work to come up with the rules for this. The two facts are:

1. The reciprocal of any number has the same sign as the number itself,
and
2. Any division problem can be written as a multiplication problem, namely the dividend times the reciprocal of the divisor.
Fact 2 above is what we used to divide fractions: 1/2 divided by 5/6 is the same as 1/2 times 6/5. Fact 2 above also tells us that 14 divided by 2 is 14 times 1/2. 65 divided by 1000 is 65 * 1/1000. Fact 2 tells us we can rewrite any division problem as a multiplication problem. And fact 1 tells us the factors in that multiplication problem will have the same signs as the dividend and divisor in our division problem. So the bottom line of it all is that we can use the same rules for division as multiplication! That is, when dividing two numbers,

like signs: positive quotient
unlike signs: negative quotient.

One consequence of the rule that's the main point of this section is that

A. –6 divided by 2 equals –3, and –6 divided by –2 equals +3,
or
B. –6 - 2=-8 and -6 +2 = -4?

424. There are some consequences of these rules that are important to keep in mind. One is that the opposite of any positive or negative number is the same as (–1) times that number. So (-1)(6)=-6, and (-1)*(-6)=6. To change any number to its opposite, we multiply by negative one.

Which of the following two questions does the rule in this section allow us to answer?

A. What would we add to -x to get x?
or
B. What would we multiply -x by to get x?

425. Here's another consequence of our rules about multiplying and dividing negative numbers. Let's suppose we are multiplying a bunch of numbers together, like these:
(-8)(2)(-4)(-7)(1/7)(-1/8).

One of our rules of multiplication said that like times like equals a positive number. So each pair of positive numbers in the string of numbers above will yield a positive number. If each negative number can pair up with another negative number, the final product will come out positive. If there is one negative number that can't pair with another negative, the final product will come out negative.

To put that in other words, if there is an EVEN number of negative numbers in our string of factors, the product will be positive. If there is an ODD number of negative numbers, the product will be negative. We don't have to count the number of positive factors, because they don't change the sign of the product.

What about when there are no negative numbers at all? 0 is an even number, so our rule holds: the product is positive.

What's a summary of the main point of this section?

A. The function y=0-x gives the opposite of whatever number is put into the function,
or
B. an even number of negative factors yields a positive product; an odd number of negative factors yields a negative product?

Chapter 20: Using Negative Numbers In Solving Equations

426. Suppose you want to solve an equation like this:

$15-x=12$

In solving equations, we seek to get x on one side, and everything else on the other side. Let's leave x on the left side of the equation. Let's subtract 15 from both sides of the equation. This leaves –x on the left side.

$-x = 12-15$

We use our rule about adding numbers of unlike signs to figure out that 12-15 is negative 3.

$-x=-3$

Now, we want to know x instead of –x. So we multiply both sides by –1. This changes the sign of both sides.

$x=3$ is our solution.

To check, we plug 3 into the original equation:

$15-x=12$ becomes $15-3=12$, and it checks.

This section gave an example of

A. an equation where there were two unknowns,
or
B. an equation where the rules about negative numbers come in handy?

427. In solving problems like the one we just solved, it's important to realize that we can think of 15-x as 15 + (-x). Thus when we subtract 15, -x is left behind. As we discussed before, we can turn any subtraction problem into an addition problem, if we have negative numbers at our disposal.

The purpose of this section was to explain

A. why you multiply both sides of the equation by –1 sometimes,
or
B. why, when you have something like 6-x on one side of an equation, you can subtract 6 and be left with –x?

428. Here's a word problem for us. There are 10 animals, some ducks and some dogs. The animals have 32 legs altogether. How many ducks are there, and how many dogs?

Let's let x=the number of ducks.
Since there are 10 animals altogether,
10-x = the number of dogs.
Since each duck has two legs,
2x= the number of duck legs.
Since each dog has 4 legs,
4(10-x)=the number of dog legs.
Since the total number of duck and dog legs = 32,
2x+4(10-x)=32.

This section

A. showed how to complete the solution of a "legs problem" once you have the equation,
or
B. showed how to arrive at the equation to solve, for a "legs problem?"

429. Now we'll continue solving the "legs problem." So far we've found out that 2x + 4(10-x)=32. In solving this equation, we'll first use the distributive law to get rid of the parentheses.

2x+40-4x=32 (We multiplied 4 by 10 to get 40, and 4 by –x to get –4x.)

Now let's combine 2x and –4x to get –2x (because 2x-4x= (2-4)x or –2x.)

-2x+40=32

Now let's subtract 40 from both sides.

$$-2x=32-40$$

Using our rule for combining numbers of unlike signs,

$$-2x=-8$$

Now we divide both sides of the equation by –2:

$x=-8/-2$.

Using our rule for dividing numbers of like sign, we find that x=4. So there are 4 ducks; since there are 10 animals there must be 6 dogs. Let's check to see if the number of legs comes out to 32: 4*2=8 duck legs, and 6*4=24 dog legs, and 24+8=32, so it checks!

This section showed how to

A. arrive at the equation for a "legs problem,"
or
B. complete the solution of the equation for a "legs problem," using some operations with negative numbers?

430. Here's another equation to solve:

$$-5=\frac{-10}{x}$$

Just as it's often useful to get rid of parentheses when solving equations, it's also often useful to get numbers out of denominators. This is especially true when x is in the denominator, because you are wanting to wind up with x=something. That sort of statement has x in the numerator of the fraction x/1, not in the denominator. Let's get rid of the denominator by multiplying both sides by x:

$$-5x=\frac{-10x}{x}$$

The x’s cancel out on the right side and leave us

$-5x=-10$

Now we divide both sides by –5.

$x=\frac{-10}{-5}$

And when we recall our rule that dividing two numbers of like sign gives a positive number, we get that

$x=2$, and we’ve found our solution.

To check, we recall the original problem

$-5=\frac{-10}{x}$

and we plug in 2 for x. Does

$-5=\frac{-10}{2}$? Yes, because of our rule that dividing numbers of unlike signs gives a negative number.

This section

A. illustrated an equation where we combined several terms with x in them,
or
B. illustrated an equation where x was in the denominator, and we got it out by multiplying both sides by x?

431. Here’s one more type of equation.

$$2x+13-7x=5-x-16$$

In solving this, you can do various steps in whatever order you want. Your goal is to combine all the x terms into one, and get it onto one side of the equation, and to combine all the “constant” terms (like 13, 5, and –16) into one, and get that on the other side of the equation. Let’s start by just simplifying on each side, using our commutative law of addition.

$-5x+13=-x-11$ (we added the 2x and the –7x on the left, and the 5 and –16 on the right).

Now let’s get rid of the –x on the right side by adding x to both sides:

$-5x+x+13=-11$ (x and –x added up to 0 on the right side.)

Now let’s combine –5x and x:

$-4x+13=-11$

Now let’s finish getting the constants on the right side by subtracting 13 from both sides:

$-4x=-24$

We isolate x on the left side by dividing both sides by –4:

$x=-24/-4$ or 6.

To check, we plug 6 into our original equation:

$2x +13 -7x =5-x-16$. If we substitute 6 for x we get

$2*6 + 13 -7*6 =? 5-6-16$. Multiplying, we get

$12+13-42=5-22$
or
$25-42=-17$
$-17=-17$

and it checks out!

This section gave an example of

A. solving an equation where you had to take the square root of both sides,
or
B. solving an equation involving several x terms (like 2x) and several constants (like 13)?

432. When we are solving equations, we use the words "terms" and "factors." It's also good to know the word "expression." What are these? An "expression" is any combination of numbers, either constants (like 5) or variables (like x).

Terms are expressions that are added to or subtracted from others. So in the expression,

$5x/y + 16x$, there are two terms: the first is $5x/y$, and the second is $16x$.

Factors are expressions that are multiplied together. So in the expression

$(6x+3)(x+1)$, there are two factors: $6x+3$, and $x+1$. Each of the factors has two terms!

The purpose of this section was to

A. show why the distributive law is often used when combining terms,
or
B. define the meanings of expressions, factors, and terms?

433. Another word that's useful to know is *coefficient.* A coefficient is a number that's multiplied by some other number. It's often a constant that's multiplied by a variable. So we say that we simplify the expression $2x+3x$ by combining the *coefficients* of x, 2 and 3, to get $5x$. (The reason we can do this is the distributive law. By the distributive law, $2x + 3x = (2+3)x$, or $5x$.)

The purpose of this section was to

A. define the distributive law,

or
B. define the word *coefficient*?

434. It's useful to know these words because they help us talk about our strategy in solving equations. Often we use the distributive law in removing parentheses by multiplying factors by one another. When we are left with a string of terms on both sides of the equation, we try to group together the terms containing x and the terms that are constants, by using the commutative law to move things closer to one another, and the golden rule of equations to subtract terms from both sides of the equation. Then we can combine the terms that contain x by adding or subtracting their coefficients, using the distributive law. We combine the constant terms by using our rules for combining positive or negative numbers. We try to get one x term on one side of the equation and one constant term on the other. Finally, we divide both sides by the coefficient of x to get an equation that reads x= our answer!

The point of this section was to

A. put into words (using the vocabulary defined in the previous sections) a typical strategy for solving an equation,
or
B. make the point that you can use the associative, commutative, and distributive laws to simplify expressions that aren't equations?

Chapter 21: Solving Inequalities

435. In the last chapter we talked about how to solve equations. Consider a problem like this:

$10x - 3 < 27$.

We read this, 10x –3 is less than 27. Now we're dealing, not with an equation, which sets two amounts equal, but with an inequality, which says that the amount on the left is less than that on the right.

There's not just one answer to this inequality. Let's guess and check, and try substituting 0 for x. We get $10*0 - 3 < 27$, or $-3 < 27$. This statement is true, so 0 works. But 0 isn't our only possible answer. Let's try substituting 1. $10*1 - 3 < 27$, when simplified, yields $7 < 27$. This is also a true statement, so 0 and 1 both work.

So what we're searching for, when solving inequalities, is a description of the whole set of numbers, each one of which makes the original statement true.

What's a major point of this section?

A. People typically represent the solutions to inequalities by darkening certain parts of a number line,

or

B. when we are solving inequalities, we are looking for a description of the set of numbers that makes the inequality true.

436. If we're given an inequality like

$10x - 3 < 27$,

how, other than guessing and checking, do we solve it? If we were solving an equation, like

$10x - 3 = 27$,

we'd first use the golden rule of equations and add the same number, 3 to both sides. We'd get

$10x = 30$.

Then we'd divide both sides by 10, and get

$x=3$.

What if we used the same rules with our inequality? We would start with $10x - 3 < 27$, and add 3 to both sides to get

$10x < 30$.

Then we'd divide both sides by 10, to get

$x<3$.

If we check out as many numbers as we want, we find that the set of all numbers less than 3 makes our original inequality correct, and the set of all numbers 3 or greater makes our original inequality incorrect. So we have come up with a correct answer! The set of all x such that x is less than 3 is it!

So far it looks as if the regular rules of solving equations also work with inequalities. But let's not get too confident! There are important exceptions!

This section

A. presented an example where the regular rules of solving equations worked with an inequality, but cautioned that there are important exceptions to those rules,
or
B. illustrated an operation that reverses the sign of an inequality?

437. Let's check out the rules of addition and subtraction with an inequality, namely

$6 > 3$.

If we add 2 to each side, we get $8 > 5$. If we add 10 to each side of the original inequality, we get $16 > 13$. if we subtract 10 from each side of the original inequality, we get $–4 > -7$. No matter what we add or subtract, the quantity on the left remains greater than the quantity on the right. In fact, the quantity on the left is exactly three more than the quantity on the right.

The same thing holds when one of the quantities is a literal number. The rule always holds, that if we add or subtract the same number from both sides of an inequality, the solutions to the new inequality will be the same as those of the old inequality. So the addition and subtraction rules for inequalities are just like those we use in solving equations.

This section says that

A. We can go ahead and add or subtract any number to or from both sides of an inequality when we are solving inequalities.
or
B. Sometimes equations have more than one solution, or no solutions.

438. Now let’s look at multiplication and division rules for inequalities. Suppose we start with

$6 > 3$,

and multiply or divide by positive numbers. Multiplying by 2, we get $12 > 6$. Dividing the original inequality by 3, we get $2 > 1$. Multiplying by 100, we get $600 > 300$. Dividing by 100, we get $.06 > .03$. It turns out that for any positive number we choose, the inequality remains the same. The same thing holds when one of the quantities is a literal number. If we multiply or divide both sides by any positive number, the solutions to the new inequality will be the same as those of the old inequality. So we can multiply and divide both sides by any positive number, just as we do with equations.

This section says that

A. As a general rule, the higher the power the literal number is raised to in an equation, the more solutions the equation will have.

or
B. When solving inequalities, we can multiply or divide both sides by any positive number.

439. But now let's see what happens when we multiply or divide by negative numbers. Let's start with

$6 > 3$. If we multiply both sides by –1, we get –6 and –3. But –6 is less than –3, not greater. So we have reversed the direction of the original inequality!

$-6 < -3$.

Suppose we multiply both sides of our original inequality by –10. Again we reverse the direction of the inequality:

$-60 < -30$.

Suppose we divide both sides of our original inequality by –3. Again we reverse the inequality sign:

$-2 < -1$.

The same holds true with expressions involving literal numbers. So our rule becomes: if we multiply or divide both sides of an inquality by a negative number, we reverse the direction of the inequality.

The major rule stated in this section is what?

A. If you square both sides of an inequality, you had better check your answer by substituting into the original inequality.
or
B. If you multiply or divide both sides of an inequality by the same negative number, you reverse the direction of the inequality sign.

440. Suppose we're out to solve an inequality such as this:

$10 – 6x < 40$.

First we subtract 10 from each side. We get

$-6x < 30$.

Now, we divide both sides by –6, and reverse the inequality sign. We get

$x > -5$.

When we try substituting various numbers in our original inequality, we see that we've come up with the correct solution set. Every number greater than –5 makes the original inequality true, and every number –5 or less makes it false.

441. When we're working with inequalities, we can always make the amounts switch places with each other and switch the direction of the sign, any time we feel like it. If $5 > 3$, then $3 < 5$. If $100 < 3000$, then $3000 > 100$. If $a < b$, then $b > a$.

Suppose that we start with the inequality

$8 > 4x$.

We divide both sides by 4, and get

$2 > x$.

But we might prefer to give our answer in terms of "all numbers x such that x is less than 2" rather than "all numbers x such that 2 is greater than x." So we can always make 2 and x switch places, and reverse the inequality sign, to come up with

$x < 2$.

What rule did this section state?

A. If you divide both sides of an inquality by the same negative number, you should reverse the inequality sign.
or

B. Any time you want, you can make the sides of an inequality switch places if you reverse the inequality sign.

442. What if you want to multiply or divide both sides of an inequality by a certain number, but you don't know ahead of time if the number is negative or positive? This is when things get tricky. Suppose you have an inequality like this:

$6/x <1.$

We'd like to multiply both sides by x. But we don't know yet whether x is positive or negative. How do we proceed? What we do is to first assume that x is positive, and see what we get. Then we assume that x is negative, and see what we get. Any answers that make sense and check out with the original inequality are part of our solution set.

So if we assume that x is positive, we multiply both sides by x and leave the inequality as it is. We get

$6 < x.$

This is the same as $x > 6$. When we look at the original inequality, we see that any number greater than 6 is going to make the inequality true. So we've found at least part of the solution set, which is the set of all x greater than 6. But we're not finished yet!

In the solution so far, we have

A. assumed that x is positive, and worked out the answer on that basis,
or
B. stopped working, because we can't know whether x is positive or negative?

443. But now let's assume x is negative. Here's our original problem:

$6/x < 1.$

We multiply both sides by x and reverse the inequality sign, because x is negative. We get

$6 > x$, or $x < 6$.

So now what we've found is that if x is negative number it will work so long as the number is less than 6. But all negative numbers are less than 6. So any negative number will make the original inequality true. When we check out the original inequality, we find that any negative number does indeed work. So our answer, combining both assumptions, is the set of all numbers less than zero or greater than 6!

The strategy that was used in solving the inequality $6/x < 1$ was what?

A. First assume x is positive, then assume x is negative, and combine the solution sets we get with both assumptions.
or
B. Avoid doing the problem, because you can not know whether x is positive or negative.

444. So far we've been working with inequalities that have a greater than sign (>) or a less than sign (<). Almost exactly the same procedures work when we have a greater than or equal to sign (like this: ≥) or a less than or equal to sign (like this: ≤).

If we start with an inequality like

$3x + 1 \leq 16$,

we can solve it just as we solved other inequalities. We get

$3x \leq 15$

and then

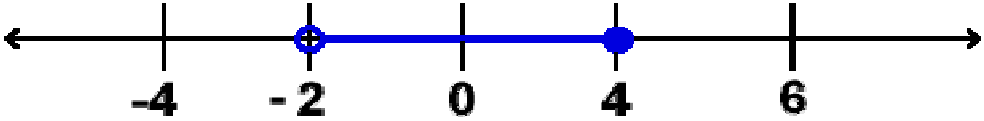
$x \leq 5$. So our answer is, the set of all x such that x is less than or equal to 5. (5 is in our solution set.)

This section

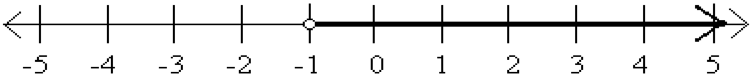
A. gave an example of an inequality using absolute values,
or
B. gave an example of an inequality using a less than or equal sign?

445. People represent the answers to inequalities in several ways. One is the description of the solution set that we have already illustrated. A second is by a picture of the number line, with the region of the solution set darkened. If the solution set goes off forever in one direction, people represent that by just drawing an arrow in that direction. If the solution set starts with a certain number and includes that number, such as the set of all x so that x is less than or equal to 5, we draw a dark circle over the 5. If the solution set doesn't include the 5, for example the set of all x so that x is less than 5, we draw a hollow circle over the 5.

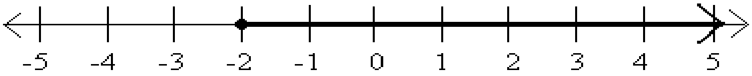
Here are some examples.



The number line above shows the set of all x such that x is greater than –2, and less than or equal to 4. Thus 4 is in that solution set, but –2 is not. Another way that people represent this is the set of all x such that $-2 < x \leq 4$. We would read this, the set of all x such that –2 is less than x, which is less than or equal to 4.



The graph above shows the set of all x such that x is greater than –1.



And the graph above shows the set of all x such that x is greater than or equal to –2.

What's a summary of the above section?

A. To represent the solution to an inequality, you darken on the number line the region in the solution set, using hollow or dark circles to represent whether the end is not included or included, respectively, and using arrows to represent the solution set going off indefinitely in a certain direction.
or
B. Inequalities involving absolute values sometimes have solution sets that go off indefinitely in both directions, but exclude some region in the middle.

446. There's another way that people communicate about the solution sets of inequalities. This is called interval notation. Above, we looked at a graph of the set of all x such that $-2<x\leq4$. We could write this set of numbers like this: (-2, 4].

The parenthesis on the left side means that the interval doesn't include –2. The bracket on the right side means that the interval does include 4.

Let's practice with another interval. What is meant by [-10, 6)? It's the set of all x such that $-10\leq x<6$. –10 is in the set, and 6 is not in the set. All the numbers between -10 and 6 are in the set.

Which of the following is an example of what this section was trying to teach?

A. If we want to use interval notation to talk about all the numbers between 5 and 15, counting 5 and 15, we would write [5,15].
or
B. If we have an inequality like $|x|>3$, the darkened part of the number line goes off indefinitely in both directions, leaving an undarkened part between –3 and 3 and a small circle over –3 and 3.

447. How do we use interval notation to represent something like the set of all x such that $x>3$? The interval goes off to the right on the number line without stopping. A graph would indicate this by an arrow. What we do when we use interval notation is to use the following symbol: ∞, which is the symbol for the word "infinity." We would write the set of all x such that $x>3$ in interval notation by writing $(3,\infty)$. This means that we can't think of a biggest number for this set, because any number that we name, we can always think of a bigger number. We could name bigger and bigger numbers forever, without ever coming to the biggest one in the set. (But we could probably think of better things to do with our time, even if we had forever!) When we use interval notation, we always put a parenthesis around the side of the interval with infinity on it. This is because the numbers go up higher and higher but never actually reach any number called "infinity."

What is a consequence of what this section tried to teach?

A. To represent the interval where $14<x\leq 28$, we would write (14,28].
or
B. To represent the interval where $x\geq 5$, we would write $[5,\infty)$.

448. What if we want to indicate a solution set that goes indefinitely toward more and more negative numbers? For example, how would we indicate the interval

where $x \leq 0$? In interval notation, we would write it like this: $(-\infty,0]$. The symbol $-\infty$ is read “negative infinity.” If we think of negative a million, and then negative a billion, and then negative a billion billion, our thoughts are moving in the direction of negative infinity (even though we never get there).

Which of the following is a consequence of what this section tried to teach?

A. Because there is no end to the number of decimal places you can have, an infinite number of numbers exist between 1 and 2.
or
B. We would represent the set of all numbers less than 10 in interval notation by writing $(-\infty, 10)$.

Chapter 22: Means and Medians

449. Suppose you rate your happiness on a scale of 0 to 10, where 0 represents the most miserable person you can imagine, and 10 represents the happiest person you can imagine. How happy have you been over the last few days?

Let's imagine that you ask two people that question, and both of them say, "I can't answer that. Some days are higher than others. It varies. I can't give one answer."

But suppose that each of them can give a rating to how happy they've been for each day. The first person can remember back 3 days, and the ratings were 7, 8, and 9. The second person can remember 7 days, and has ratings of 3, 1, 6, 2, 5, 8, and 3.

It looks like the typical or average day for the first person is happier than that for the second, doesn't it? But what do we mean by a "typical or average day?" People made up means and medians to try to answer this question.

The main idea of this is that

A. people are seldom exactly as happy one day as the day before,
or
B. means and medians were invented to answer the question, "what is average or typical?"

450. How can we compare these two people? Who has been happier lately? One way that might occur to us is simply to add up the units of happiness they've both had. The first person has had 7+8+9 or 24 units of happiness. The second has had a total of 28 units of happiness. So, someone might conclude, the second is the happier person.

But wait, someone else might say. We're counting more days for the second person. If we want to compare them best, shouldn't we figure out how much happiness each of them has per day?

The first one had 24 units of happiness in 3 days. The second one has had 28 units in 7 days. What if all those units of happiness that each of them had experienced were divided evenly among the days? For the first person, the 24 units would have been divided evenly among 3 days, so the first one would have had 8 units per day. For the second one, 28 units would have been divided evenly among 7 days, so the second would have had only 4 units per day.

Figuring out what would have happened if the total had been spread out evenly among all the days is called taking an average, or a mean.

This section defined an average or mean as the answer to what question?

A. What if the total of all the days' ratings were spread out evenly over all the days?
or
B. Which value is in the middle?

451. The answer to the question, "What if the total were spread out evenly," is the *average* or *mean* for any set of numbers. Suppose there are three people. They weigh 50 pounds, 100 pounds, and 150 pounds. What is their average weight?

Altogether, these people weigh 50+100+150 or 300 pounds. What if that total weight were spread out evenly among the 3 people? Then each person would weigh 300/3 or 100 pounds. The average or mean weight is 100 pounds.

What if you buy three things; one costs $12, one costs $20, and one costs $4. What is their average cost, or mean cost? Their total cost is 12+20+4 or $36. If that total cost of $36 were spread out evenly over the three purchases, each purchase would be $36/3 or $12.

So to get the mean or average of any set of numbers, you add up the numbers, and then you divide by how many numbers there were.

What's a good summary of how you find out the answer to the question, "What if the total were spread out evenly?" so as to find the mean of a set of numbers?

A. If you have a certain set of numbers, and you make those numbers bigger, you increase the total of them.
or
B. To find the mean or average, you find the total of the numbers and divide by how many numbers there are.

452. The phrase "how many numbers there are" is interesting. We could also call it the "number of numbers." There are times when you count things several times, to get a bunch of numbers, and then you count how many times you counted! To keep from getting too confused about this idea of "the number of numbers," people sometimes give it a name. The bunch of numbers is called the sample, and

how many numbers there are is called the size of the sample, or the *sample size*. So if we use this term, we say that the mean of any set of numbers is the total of those numbers divided by the sample size.

The purpose of this section is to

A. define the phrase *sample size*,
or
B. let you know that you can't have an average of 0 numbers?

453. If someone gives you a bunch of numbers and asks you to find the mean, you first find the total and then divide by the sample size. What if you are asked to go in the other direction? What if someone gives you the mean and the sample size, and asks you for the total of the numbers? For example: the mean weight of 10 people is 140 pounds. How much would the 10 people weigh altogether?

If mean=total/sample size,

then we can multiply both sides of this equation by the sample size. We get

mean * sample size = total.

So in the problem, we would multiply the 140 pounds by 10 and get that the total weight of all ten people was 1400 pounds.

The purpose of this section was to tell you how to

A. find the average by dividing the total by the sample size,
or
B. find the total by multiplying the average by the sample size?

454. Here's a type of problem that test-makers seem to like. Suppose the average weight of four people is 100 pounds. Then a fifth person joins the group. Now the average weight of all 5 people is 110 pounds. How much did the fifth person weigh?

In the last section we talked about how the mean times the sample size equals the sum or total for the group. Let's use that idea to solve the problem.

We’re told that the average weight for 4 people is 100 pounds. That means that the total weight of the four people is 400 pounds. We’re also told that the average weight of the group of five people, after the fifth person joined, was 110 pounds; that means the total weight for all 5 people was 5*110, or 550 pounds.

Now we have a simpler problem. The group weighed a total of 400 pounds before the fifth person came in. After the fifth person came in, adding some more weight, the group weighed a total of 550 pounds. How much did the fifth person weigh? This problem can be solved by subtraction: 550-400=150. The fifth person must have weighed 150 pounds.

The most important idea in solving the problem that this section explained was that

A. the total of a set of numbers is their average, multiplied by the sample size, or
B. the average can be greatly affected by a single value that’s very large or very small?

455. The mean is one way of answering the question, “What’s a typical value for a set of numbers?” A different way of answering this question is by figuring out the *median* of the numbers. If you put a bunch of numbers in order, from least to greatest, the value that is in the middle is the median. Do you remember our imaginary person who on 7 days had happiness ratings of 3, 1, 6, 2, 5, 8, and 3? Let’s put them in order. In order, the 7 ratings are: 1, 2, 3, 3, 5, 6, 8. There are 7 numbers, so the fourth number is the median. That number is 3. You’ll notice that we include 3 twice because it was observed twice.

What do we do if there is an even number of numbers? For example, how do we find the median of 2, 3, 4, and 6? There are two middle numbers in this set, namely 3 and 4. When there are two middle numbers, the median is defined as the number halfway between the two. So the median of 2, 3, 4, and 6 is 3.5. You can get the number halfway between two numbers by taking the mean of those numbers. So when you have an even number of numbers, the median is the mean of the middle two numbers.

What’s a summary of the point of this section?

A. For an odd number of numbers, the median is the middle number when all are arranged in order; for a set with an even number of numbers in it, the median is the mean of the two middle numbers when they're arranged in order.
or
B. You need to add and divide when finding the mean.

456. How should we think about what the median of a bunch of numbers means? Here's one way. Suppose that we wrote all the numbers in the set on cards and put them in a hat (or an urn, or a box, or any other container you choose)! Then we drew out one number at random. The chance that the number we picked is above the median is just the same as the chance that it was below the median. Thus the median is the number that forms the dividing line. Whenever we want our measure of "typical" to be the number that divides the data set into two parts in this way, the median is what we want to use.

The point of this section is that

A. if we pick a number from a set at random, there's an equal chance that the number will be above or below the median,
or
B. the 60th percentile of a set of numbers is the point where about 60 percent are below that number, and 40 percent are above it?

457. When you do have one or a few numbers that are lots bigger or smaller than the rest, those numbers are called *outliers*. For example, suppose you have the following set of numbers: 2, 3, 6, 4, 8, 1000, 4, 1, 5. One of them is not like the others, isn't it? The number 1000 is an outlier in this set. On the other hand, consider the following set of numbers: 999, 1006, 1001, 991, 1010, 2. Now the number that's obviously an outlier is 2. The number 1000 would not be unusual at all in this group.

How much different does a number have to be from the others before it qualifies as an outlier? There are various rules that people have made up. Let's not concern ourselves with them now, but just imagine plotting a bunch of numbers on the number line, and say that the farther one or a few numbers are from all the rest of them, the more likely those far-away numbers are to be called outliers.

The point of this section is that

A. The first quartile is the number greater than one-fourth of all the numbers, and the third quartile is the number greater than three fourths of the numbers.
or
B. The farther one or a few numbers are from all the rest of them, the more likely those far-away numbers are to be called outliers.

458. How do the mean and the median relate to each other? When the numbers are fairly evenly and symmetrically spread out, without having outliers, the mean and median give fairly similar results. For example, for the person with the seven happiness ratings of 3, 1, 6, 2, 5, 8, and 3, let's see how the mean and median compare. We've found that the median is 3. We've also mentioned that the person had a total of 28 units of happiness, spread out over 7 days, so his mean level of happiness was 4 units. With a median of 3 and a mean of 4, the results are not very far apart.

The point of this section is that

A. means and medians are both very frequently used in the field of statistics,
or
B. when the numbers in the set are spread out symmetrically, without outliers, the mean and the median usually are pretty close to each other?

459. What's the median of the numbers 1, 2, and 3? Since they're in order, and 2 is in the middle, 2 is the median. What's the mean? Since they add up to 6, and there are 3 numbers, and 6 divided by 3 is 2, 2 is the mean also.

Now let's think about what happens when there are outliers. Suppose that instead of 1, 2, and 3, we have 1, 2, and 1000. Relative to 1 and 2, 1000 is an outlier. The median is still 2, because it's still in the middle. The mean is 1003 divided by 3, which is a little over 334. The outlier had a very big effect on the mean, whereas it had no effect at all on the median.

The general rule that is the main point of this section is that

A. the mean is influenced much more strongly by outliers than the median is,

or

B. the mean is calculated by adding up the numbers and then dividing by how many numbers you have?

460. So which is a “better” measure of what’s typical of the data set: the median, or the mean? It depends upon whether we want a measure of what we would get if the total were evenly divided, or the number that splits the ordered numbers in two! Let’s think of an example of each.

Suppose we have two tug of war teams. We measure how much force each player can pull with. These are measures of how strong the team members are. For team A, the 4 members have strengths of 100, 102, 104, and 1000. The last player on team A is an outlier, unusually strong. For team B, the members have strengths of 150, 200, 204, and 206. The median strength for Team A is 103, whereas for Team B it’s 203. Team B looks to be much stronger. But for Team A, the mean strength is about 327, whereas for Team B, the mean strength is about 190.5. The mean strength is greatly influenced by the outlier, whereas the median strength is not. Which measure is better?

For tug of war, what’s important is the total strength of the team – the total force that the team is able to pull with. In predicting who wins, the mean is much more effective than the median, and is the “better” measure in this case.

This section gives an example of a situation in which

A. because the total of the numbers in the set was what’s important, the mean was a better measure than the median,

or

B. the presence of an outlier made the mean not useful as a measure?

461. On the other hand, let’s imagine a different situation. Let’s suppose there are two plans to raise the income of people in different parts of a town. Both towns start out with everyone making $10,000. In the first town, everyone in the whole town doubles their income, and now earns $20,000 per year. In the second town, everyone stays exactly the same, except for one person who wins the lottery and has an income of 50 million dollars. It turns out that the average income in that town is $30,000 per year.

Now which measure is a better measure of whether the income-raising plans worked? The median for the first town doubled, whereas the median for the second town stayed the same. On the other hand, the mean for the first town doubled, but the mean for the second town tripled! In this case, the median is the better measure. In this case we want a measure that isn't greatly influenced by outliers.

This section gives an example of a situation where

A. the median is a better measure than the mean, because it is easier to calculate,
or
B. the median is a better measure than the mean, because it isn't so influenced by an outlier?

462. Thus sometimes the mean is a better measure, and sometimes the median is better. This is the reason why math teachers ask students to learn about both of them!

Before leaving this section on statistics, let's talk about percentiles. Let's imagine that people compete on a test, trying to "beat" as many other people as they can. The percentile answers the question, "What percent of the test-takers did you beat?" So if you scored in the 60th percentile, that means that you scored higher than 60% of the test-takers. If you scored in the 99th percentile, you scored higher than almost everybody, and if you scored in the 1st percentile, almost everybody else scored higher than you.

What about ties? There are different rules for scoring percentiles when scores are tied. Sometimes percentiles mean "What percent of people did you beat," and sometimes they mean "What percent of people did you either beat or tie?"

The main point of this section is that

A. when you are making up a test, you want to have it long enough that there are not many ties,
or
B. the percentile of a certain number in a set means, "What percent of the numbers was this number greater than," or sometimes "What percent of the numbers was this number greater than or equal to?"

463. If you want to be in the top 10% of your class, what percentile should you aim for? You should be in at least the 90th percentile. If you want to be in the top quarter of the class, you aim for the 75th percentile. If someone's income is exactly at the median for the country, at what percentile is her income level? The median is the level higher than about half, or 50% of the people in the country, so the answer is the 50th percentile. The 25th percentile is referred to as the first quartile, and the 75th percentile is referred to as the third quartile.

The point of this section is to

A. give several examples of how people speak about percentiles,
or
B. prove a mathematical theorem about percentiles?

Chapter 23: The Fundamental Counting Principle

464. How many ways can something happen? For example, how many ways can I choose what to wear? I have two colors of pants: black, and tan. I have three colors of shirts: blue, gray, and pink. How many different ways can I be dressed? (We're disregarding my other items of clothing.)

One way to answer this question is to simply list all the possibilities, and count them. Let's do that with my pants and shirts:

pants	shirt
1. black	blue
2. black	gray
3. black	pink
4. tan	blue
5. tan	gray
6. tan	pink

When we count these, we get a grand total of six different possible outfits. We don't even need to use our fingers to count them, because I numbered them.

This section

A. gave a formula for the "fundamental counting principle,"
or
B. illustrated how you can count the number of ways something happens by making a list and simply counting the items on your list?

465. What if I had had 10 different colors of shirts, and 5 different colors of pants? Would we really need to a) list all ten colors of shirts for the first color of pants, b) then list all ten colors of shirts for the second color of pants, c) keep going until we've listed the ten colors of shirts 5 times, and d) then count the items on the list? Not unless we like doing unnecessary work! As we made such a list, we would quickly realize that we were writing down ten things, five different times, and this is just a multiplication problem! We wouldn't bother to write down

all 50 combinations. We would simply multiply 5 by 10 and say that there were 50 different possibilities for my colors of pants and shirts!

This lazier and more efficient way of answering this question is called the fundamental counting principle. It says: to get the number of ways of making two choices, we take the number of options for the first choice, and the number of options for the second choice, and multiply them together!

The fundamental counting principle implies which of these two?

A. If there are 5 sandwiches on the menu, and 3 types of drinks, there are 5 * 3 or 15 ways of choosing a sandwich and a drink together.
or
B. If the sum of 4 numbers is 10, and the sum of the numbers after you add a fifth one to the sample is 15, the fifth number must have been 5.

466. What if we take into account that I have two colors of shoes, black and white, in addition to the two colors of pants and 3 colors of shirts? What if we now want to know how many different ways I can select a combination, not of two types of clothes, but three types? If we made a list, we would list the six combinations of pants and shirts that we listed above, twice: once to go with white shoes and once to go with black shoes. So we would have a total of 12 possibilities, which is what we get by multiplying 2 types of pants * 3 types of shirts * 2 types of shoes. This example illustrates that we can use the fundamental counting principle when we are counting combinations not just of 2 types of things, but of 3 or more as well.

The main point of this section is that

A. if you take into account 5 types of shoes, you would multiply the number of outfits by five,
or
B. the fundamental counting principle works when you are counting combinations not just of two things, but of any number of things?

467. Here's another example of how to use the fundamental counting principle. A car manufacturing plant makes a certain model of car with 3 different types of seats and interiors (leather, fake leather, and fabric); with 2 different types of gear

shifts (automatic transmission and non-automatic transmission); and with 7 colors (red, orange, yellow, green, blue, indigo, and violet). How many different types of cars does the plant make, for this model, if it makes all possible combinations? We would just multiply 3*2*7 and get 42 different types of cars.

What was the purpose of this section?

A. To give another straightforward example of how the fundamental counting principle is used.
or
B. To show how the fundamental counting principle is used in figuring out probabilities.

468. Here's a problem where it isn't so straightforward how to use the fundamental counting principle. Suppose I have a penny, a nickel, and a dime. How many different amounts of money can I give you, with those coins (including giving you 0 cents)? We could make a list, like this, as to whether I give you each coin. To keep things straight, I'll first list 0 coins given, then 1 coin, then 2 coins, then 3 coins.

Penny	Nickel	Dime	Total Given
No	No	No	0 cents
Yes	No	No	1 cent
No	Yes	No	5 cents
No	No	Yes	10 cents
Yes	Yes	No	6 cents
Yes	No	Yes	11 cents
No	Yes	Yes	15 cents
Yes	Yes	Yes	16 cents

We count up the total number of outcomes on our list, and we get that there are 8 of them. So the answer is 8.
But making lists like this gets tedious once you've done a few. There's a simpler way to do this problem.
We think of three choices: one about the penny, one about the nickel, and one about the dime. There are two options for each: give it to the other person, or not. So the total number of ways we can make choices for all three things, using the

fundamental counting principle, is 2*2*2, or 8. This is the result we got from our list!

The purpose of this section was to

A. show how the fundamental counting principle is used in problems other than the "how many outfits" or "how many different menu choices" types of problems,
or
B. show how the fundamental counting principle is used to figure out how many different ways some things can be arranged in order?

469. Our problem with the penny, nickel, and dime is a special case of the more general problem of "how many subsets." You recall that if you have a certain number of things, and you pick out some (or none, or all) of them, you are picking a subset. The answer to the question, "How many subsets," is 2 multiplied together to make as many factors as there are elements in the set. With the penny, nickel, and dime, we had 3 choices, and two options for each. So we had 2*2*2 possible subsets. If someone asked, how many different subsets can you make of the first 10 letters of the alphabet, we could think of 10 different choice points, and 2 options for each. So the answer would be:

2*2*2*2*2*2*2*2*2*2,

or two multiplied by itself for a total of 10 factors. (In a future chapter, we'll discuss that a shorter way of writing this is 2^{10}, which is called 2 to the 10th power.)

A fact included in this section is which of the following?

A. The word *permutation* means an arrangement where order is important, and the word *combination* means one where order doesn't matter?
or
B. When you want to know how many subsets you can make of n things, the answer is 2 multiplied together for a total of n factors.

470. Have you ever seen puzzles where you are given the letters of a word, in a scrambled-up order, and your task is to figure out the word? Let's suppose you

had a computer program that would give you every possible order that the letters could be in. How many different orders would be possible, let's say for a 5 letter word? Let's imagine that the letters, scrambled, are voaeb. We can solve the question of how many different orders by the fundamental counting principle, and it turns out that this is a very important question to ask in lots of situations involving probability.

The point of this section is which of the following?

A. In some situations, the order in which objects are arranged is important, as in how people place in a contest; in other situations, the order is unimportant, as in which people get selected to be on a committee.
or
B. We can use the fundamental counting principle to figure how many ways we can arrange things (such as letters) in different orders.

471. Here's how we solve the problem of how many different orders for the letters voaeb. We have five different positions to fill in our word. For the first letter, there are 5 choices, because we have 5 letters. But now we go to the second position. How many choices do we have for which letter will go second? It isn't five, because we've already used up one letter for the first slot, and we can't repeat a letter. So we only have four choices left. For the third letter, we have three choices left; for the fourth letter we have two choices left, and for the fifth letter, we have only one choice left. So when we do our multiplication, it is like this:

5 * 4 * 3 * 2 * 1

which comes out to 120. So there are 120 possible ways of rearranging those 5 letters. Thus it's pretty amazing when someone can study the letters voaeb for a few seconds and come up with the right answer – the word "above."

This section illustrated which of the following facts?

A. When we are figuring out how many ways we can arrange a certain number of things in order, we start with the number of things there are and multiply by a number that is one less each time, until we get down to 1?

or
B. Solving certain types of puzzles may exercise the brain in ways that helps it solve other problems that come up in real life.

472. Here's a very similar problem. Suppose there are 10 runners in a race. Assuming that there are no ties, and that they all finish, in how many different orders can the runners cross the finish line? There are 10 ways to pick who will come in first. But once someone has come in first, there are only 9 left to pick from, for second place. And so on, so the answer becomes

10 * 9 * 8 * 7 * 6 * 5 * 4 * 3 * 2 * 1

which comes out to over 3 million ways!

This section

A. gives another example of how you use the counting principle to figure out how to arrange a certain number of things in order,
or
B. gives examples of the types of selections where order doesn't make a difference, such as selecting 5 people to go on a trip out of a group of 20 people?

473. Mathematicians come up with expressions like 5*4*3*2*1 and so forth often enough that they have decided on a shorter way of writing them and speaking about them. 5*4*3*2*1 is called "5 factorial" and is written "5!" The exclamation point is what means "factorial," or "start with the number and multiply by factors going down one each time till you get to one."

Which of the following is an example of main message of this section?

A. An expression like 3*3*3*3 is called "three to the fourth power" and written 3^4,
or
B. An expression like 4*3*2*1 is called "4 factorial" and written 4!

474. Here's a similar problem. Suppose we have a club with 10 people in it. How many ways can we pick a president, vice-president, and secretary? (This club doesn't need a treasurer because it doesn't collect or spend any money!)

Suppose we pick the president first. There are 10 ways to make this choice. Next, we choose the vice president. There are 9 people left to choose from. Next, there are 8 people left to choose from for secretary. So the total number of ways we can make the choice is

10 * 9 * 8, or 720.

This problem illustrates the way that you figure out

A. how to compute the number of ways of picking three specific officers from a set of 10 club members,
or
B. how to compute the number of ways of picking three people from a set of 10, to go on a trip or be on a subcommittee, when it doesn't matter who is first, second, or third?

475. In the problem we just did, we say that "order makes a difference," because Joe-president, Jean-vice-president, and Susan-secretary, is a different choice than Jean-president, Susan-vice-president, and Joe-secretary. The order to which the three people are assigned to the three posts makes a difference to them, and to the club members.

But here's a new problem: what if we don't care what order the people are in? What if we are just picking three out of 10 people to go on a trip, and it doesn't matter who is number one, number two, and number 3? How many ways can we do that?

We already know that if we were to call the people president, vice-president, and secretary, or person 1, person 2, and person 3, there would be 720 ways of picking them. Our strategy for solving the new problem, which is finding the number of unordered groups possible, is to start with this 720 ordered groups, and to figure out by how much the number 720 overestimates the number of unordered groups we can pick.

What sort of problem are we working on now?

A. One where we want to know how many ways there are of picking 3 people from 10, where order makes a difference, for example one goes first, one goes second, and one goes third?
or
B. One where we want to know how many ways there are of picking 3 people from 10, where order makes no difference, for example where 3 people are going to get to go on a trip?

476. Let’s suppose that the group of people we pick are Joe, Jean, and Susan. There are six different ordered outcomes that will result in this group:

1. Joe Jean Susan
2. Joe Susan Jean
3. Jean Joe Susan
4. Jean Susan Joe
5. Susan Joe Jean
6. Susan Jean Joe.

Instead of making a list, we could have used our counting principle to figure out how many ways to rearrange 3 people. The answer is 3*2*1, or three factorial, or 6, the same number that we got by listiing.

So here’s the tricky point. If we don’t care what order they are in, we think of each of the above 6 ordered groups as the same unordered group. All of those six were just one way of picking 3 people to go on the trip. So in computing the 720 ordered ways of picking 3 people, we got six times as many groups as we are really interested in, if order makes no difference. So to get the number of unordered groups, we should divide 720 by 6, and get 120. We’ve found the answer!

This section

A. showed how to compute the number of ways you can choose a certain number from a larger group, if order makes no difference?
or
B. showed how to find the total number of subsets you can make, of a certain number of things?

477. Sets of things where order makes a difference are called *permutations*, and sets where order doesn't make a difference are called *combinations*. So if I'm talking about a group of letters and I consider b, c, and a the same group as a, b, and c, then I'm talking about a combination of letters. But I consider b,c,a to be one arrangment, and a,b,c to be a different arrangement, then I'm referring to a permutation of letters.

You've probably seen a lock where you have to dial three numbers in order to get the lock open – for example, 16-35-8. For any such locks that I've seen, order is important. If 16-35-8 works, 8-35-16 would *not* work. So if we want to use words consistently, we should call such locks "permutation locks." Instead, most people call them "combination locks." Whoever named them probably didn't ask a mathematician what to call them!

The main purpose of this section was to

A. tell you the meaning of the words *permutation* and *combination*,
or
B. complain to you about what certain locks are called?

Chapter 24: Probability

478. Even in a seeming exact science like mathematics, people still debate how we should think about the meaning of probability. Some people feel that probability is a way of putting numbers on how sure we are that something will happen, or will prove to be true. If we feel totally and completely convinced that the earth will not explode today, we say that the probability that it will explode is 0. If we feel totally convinced that some life on Earth will continue at least until tomorrow, we say that the probability of that is 1, or 100%. If we feel that two possibilities are equally likely, such as that a flipped coin will come up heads or tails, we say that the probability of heads is ½ or 50%, and the probability of tails is also ½, or 50%. Probabilities can take on values from 0 to 1, or 0% to 100%, and the higher the number, the more likely we feel something is to prove true.

One of the main ideas so far is that

A. probabilities get bigger the more convinced we are that something will prove true, and they go from 0 (no chance at all) to 1 (total certainty),
or
B. there are rules that say when you multiply probabilities and when you add them, to get the probabilities of more complex events?

479. Games of chance and gambling correspond in lots of ways to the situations in life that force us to decide what to do even though we are uncertain how things will turn out. Let's suppose that there is an election where Smith and Jones are running for president of an organization. Suppose I say, "Let's bet on who will win. We both know that Smith is much more likely to win. I'll put $1 into the pot, and you put $9. If Jones wins, I get the $10 in the pot, and if Smith wins, you get the $10." Suppose we both agree that this is a fair bet, favoring neither of us. In that case, we both think that the probability that Smith will win is 9/10, or 90%. We think it's reasonable that you should risk 9 times as much as I risk, because you're 9 times more likely to win. The bet is fair, because if it were repeated over and over, my winning $9 one-tenth of the time would balance out your winning $1 nine-tenths of the time. Some people believe that imaginary bets like this are really the best way to define what we mean by probability.

This section explains that

A. probability may be thought of in terms of how frequently something happens in the long run,
or
B. probability may be thought of by imagining gambling situations, and thinking about what sort of bet would not give an advantage to either gambler?

480. Other people prefer to think of probability in terms of the frequency with which something happens if you have many, many trials. For example, if we keep careful records and find that it rains in a certain location on 12% of days, the probability of rain on any given day is 12%, or .12. (This assumes that we don't know anything else about this particular day, for example that there are big black clouds overhead at the moment.) If we find that 4 out of a thousand people get seizures when they take a certain drug, we say that the probability that someone starting the drug will get seizures is 4/1000, or .004. (Again, this assumes that we don't know anything else about this particular person, such as the fact that the person has had symptoms in the past that could have been due to seizures.)

This section illustrates thinking of probability in terms of

A. how frequently something happens, when it has the opportunity to happen many, many times,
or
B. how many ways you can have a success, divided by how many ways the outcome can occur at all?

481. One of the most useful ways to think about probability when solving certain types of test problems is to imagine a number of equally likely outcomes of an experiment of some sort. An example of an "experiment" is reaching into a box and drawing out a ball of a certain color. Each ball is equally likely to be drawn. Then the probability of a red ball, for example, is equal to the number of red balls divided by the total number of balls in the box. If there are 4 red balls and 10 balls altogether, the chance of drawing a red ball is 4/10, or 40%.

Here are a couple more examples. If we roll a cube-shaped die, where the faces are numbered 1 through 6, and each side is equally likely to come facing up, then we say that the chance of rolling a 1 (or a 2, or a 3, and so forth) is 1 out of 6, or 1/6. If we draw a card at random from a regular deck of cards that has 13 each of spades, hearts, diamonds, and clubs, for a total of 52 cards, the chance that we will draw a heart is 13/52. The fraction 13/52 reduces to ¼, so we could also say that the chance of drawing a heart is 1 in 4 or 25%.

This section focuses on probability as

A. a function of how much we would think it fair to stake on a bet,
or
B. how many ways an experiment can give a certain outcome, as a fraction of how many possible outcomes there are, when all outcomes are equally likely?

482. Here's a sample probability problem using the last notion we talked about: how many ways can an experiment come out a certain way, as a fraction of how many outcomes there are. The problem: There is a box with 4 red poker chips, 3 blue chips, and 3 white chips. If you mix them up thoroughly and draw one out at random, what's the probability of drawing a red chip?

How many ways of drawing a red chip are there? There are 3 ways, one for each of the three red chips in the box. How many total outcomes are there for the experiment? If we add 4+3+3 we get that there are 10 chips altogether, so there are 10 ways that we can draw a chip. So the probability of drawing a red chip is 4/10, or 40%.

The principle that the solution to this problem uses is that

A. the higher the odds we would be willing to give in what we think is a fair bet, the higher we think is the probability that something will happen,
or
B. the probability of an event is the number of ways that event can happen in an experiment, divided by the total number of ways any outcome can happen, assuming all outcomes are equally likely?

483. Sometimes we think of a certain overall probability that something will happen, but this probability is changed when we are given some more information. For example, the probability that it will rain on a certain day in a certain imaginary city is 12%. But the probability that it will rain on a certain day, given that it is the time of year called the "monsoon season," is 95%. The new

probability of rain given the condition of monsoon season is called a conditional probability.

The point of this section was to

A. give an example of when you add probabilities,
or
B. give an example of conditional probability?

484. Much very important information can be expressed in terms of conditional probabilities. A 2006 European study quoted on the Internet found that the probability of developing lung cancer was 0.2% for males who never smoked, and 0.4% for females who never smoked. But for males who smoked more than 5 cigarettes per day, the probability of getting lung cancer in a lifetime is 24.4% and for females the probability is 18.5%. If you are trying to decide whether or not to smoke, those differences in conditional probability are very important, aren't they?

The point of this section is that

A. much of the important information we need to know for making decisions can be expressed by conditional probabilities,
or
B. lung cancer has decreased in the U.S. over the last few decades as smoking rates have fallen?

485. Suppose someone thinks, "Lung cancer is not such a big deal. I'll go to a doctor and get medicine for it which will make me better." Do you think conditional probabilities would be useful in thinking about whether this is true? According to statistics I read, the chance of dying within 5 years, given that one has just been diagnosed with lung cancer, is about 87%. By comparison, the conditional probability of dying within 5 years given that you're a woman who has just received a diagnosis of breast cancer, is about 20%. These conditional probabilities give more precision to statements like "Lung cancer is a very dangerous disease, and few are cured from it."

The previous section gave information on the probability of lung cancer given smoking or nonsmoking. This section gave information on the probability of

A. breast cancer given smoking or nonsmoking,
or
B. death within five years, given lung cancer or breast cancer?

486. Whenever you decide that one option is better than another, you do it because you guess that the conditional probability of something good happening is greater given that option than given a different option. For some examples:

I get a job because the conditional probability of my having enough money is much higher given that I'm working than if I'm not.

I brush my teeth because the conditional probability of getting cavities and gum disease is much lower given that I clean my teeth than if I don't.

I speak kindly to my spouse because I want to increase the probability of having a happy marriage.

I study a test prep book because I want to increase the probability of a high score on a test.

I wear a seat belt when I'm in a car because the conditional probability of being hurt badly or killed in a car accident is less with a seat belt than without it.

The point of this section is which of these two?

A. Conditional probabilities are very important for decisions – in fact we decide almost everything we decide, because we feel that the option we're choosing increases the conditional probability of something good happening.
or
B. When you have two non-overlapping outcomes of an experiment, the probability of getting one or the other of them is the sum of their individual probabilities.

487. There's a rule of probability that I want to tell you about, but first we have to define what's meant by "mutually exclusive" or "non-overlapping" events. These terms mean that if the first happens, the second can't happen; also if the second happens, the first can't happen. Each one "excludes" the possibility that the other one will occur.

For some examples, if I draw one card from a standard deck, drawing a heart and drawing a spade are mutually exclusive, because no card is both a heart and a spade. If I'm rolling a die, rolling a 1 and rolling a 6 are mutually exclusive. If we look at the weather in a given city on a certain day, getting rain and getting hail are not mutually exclusive – we could get both of them. If we draw cards from a standard deck, drawing a 4 and drawing a heart are not mutually exclusive – we could get both of these, namely by drawing the 4 of hearts.

The purpose of this section was to

A. show why the probability of something happening plus the probability of its not happening have to add to one,
or
B. define what is meant by two events being mutually exclusive or non-overlapping?

488. Suppose you are rolling a die. The probability of rolling a 1 is 1/6. The probability of rolling a 2 is 1/6. What's the chance of rolling either a 1 or a 2?

We can figure that out using the idea that probability is the number of ways an event can happen, divided by the total number of ways the experiment can come out. When we roll a die, there are two ways of rolling either a one or a two. There are six ways the experiment can come out. So the probability of either a 1 or a 2 is 2/6, or 1/3.

But there's another rule that you could have used to get the same answer. When you have two non-overlapping (or mutually exclusive) outcomes of an experiment, the chance of getting one OR the other is found by adding their individual probabilities together. The probability of rolling a 1 is 1/6, and the probability of rolling a 2 is 1/6, so the probability of rolling either a 1 or a 2 is 1/6 plus 1/6, or 2/6 or 1/3.

The rule that this section states is that

A. You add probabilities when you want to find the probability that one of two events will occur, when those events are non-overlapping outcomes of an experiment.
or
B. You multiply probabilities when you want to find the probability that both of two events will occur, when those events are independent.

489. When we do an experiment, the probability that any certain outcome will happen, plus the probability that it will not happen, have to add up to 1. An equivalent rule is that the probability that something won't happen is 1 minus the probability that it will happen. Rather than trying to prove this, let's look at some examples.

Let's look at an example. Suppose the probability of rolling a 1 when I roll a die is 1/6. Our rule says that the probability of not rolling a 1 has to be 5/6, so that the two probabilities add up to 1. It makes sense, because there are 5 other equally likely outcomes.

Suppose there are 11 poker chips in a box, 3 of which are red. If we draw one out at random, the chance of drawing a red chip is 3/11. The chance of drawing something other than a red chip is 1 – 3/11, or 8/11. This make sense, because the other 8 chips had to be something other than red.

Suppose we are told that with a certain type of cancer, our chance of living for 5 more years is 40%. This means that the chance of dying before 5 years are up is 60%, because one or the other events has to occur, and the two probabilities have to add to 1 (or 100%).

What's another illustration of the rule we stated in this section?

A. If the probability of heads on one coin flip is ½, and the probability of heads on the second flip is ½, the probability of heads on both flips is ½ * ½, or ¼.
or
B. If the probability of heads on a coin flip is ½, the probability of not getting heads on the same flip is 1 minus ½, or ½ also.

490. When you have two events that don't affect each other at all, those events are said to be independent. For example, if you flip a coin twice, the outcome of the first coin flip has no effect on the outcome of the second, and vice versa. There's a rule about computing probabilities for two or more independent events that is

worth knowing. When two events are independent, the probability that both of them will occur is obtained by multiplying together the probabilities that each event will occur.

So for example, if you flip a coin twice, the probability of getting heads the first time is ½, and the probability of getting heads the second time is ½. The probability of getting heads on both flips is ½ * ½, or ¼.

For another example, if you roll two dice, the probability of getting a 1 on the first die is 1/6, as is the probability of getting a 1 on the second die. The probability that both of the dice will come up 1 is 1/6 * 1/6, or 1/36.

What's a summary of the rule this section stated?

A. The probability that an event will not occur is 1 minus the probability that it will occur.
or
B. The probability that both of two independent events will occur is the product of their individual probabilities.

491. Now let's look at a harder problem that uses some of the rules that we've mentioned. Suppose that someone rolls 4 dice. What is the probability that at least two of them will show the same number?

It's a lot easier to answer this question if we find the probability that the event will NOT occur, and then subtract that probability from 1. If it's not true that at least two of the dice are the same, then all four of them have to be different. Let's compute the probability that all four dice come up different. The first die can come out showing any number. The chance that the second die is different from the first is 5/6, because there are 5 numbers that haven't showed up yet. The chance that the third die is different from the first two is 4/6, and the chance that the fourth die is different from the first three is 3/6. For them all to be different, all these independent events have to take place, so we multiply the probabilities together. We get 5/18. Now we have to remember that this is not the answer to the original question. 5/18 is the probability that they will all be different; 1-5/18, or 13/18, is the probability that at least two of them will be the same.

Which two rules did this problem use?

Chapter 24: Probability

A. The probability that something will occur is one minus the probability that it won’t occur, and the probability of two or more independent events is found by multiplying their individual probabilities.

or

B. The probability that one or the other of two mutually exclusive outcomes will occur is found by adding their individual probabilities, and the probability of something that definitely will not happen is 0.

Chapter 25: Exponents

492. We have said before that multiplication is repeated addition. Rather than having to write 3+3+3+3+3, we can simply write 5 x 3.

What do we call repeated multiplication? Is there a similar short cut when we want to speak of 3 x 3 x 3 x 3 x 3? Yes, there is: repeated multiplication is called exponentiation. We can write 3 x 3 x 3 x 3 x 3, or 3 multiplied together 5 times, as 3^5, or three to the fifth power. The three is called the base, and the 5 is called the exponent. People speak of "raising" three to the fifth power.

Here are some more examples: 3 x 3 x 3 is 3^3, or 3 to the third power. 5 x 5 is 5^2, or 5 to the second power.

What is 5^2? It's 5 x 5, or 25. What is 3^3? It's 3 x 3 x 3. Three times three is nine, and nine times three is 27. So three to the third power is 27. What's 2^4? It's 2 x 2 x 2 x 2. Let's start from left and go to the right: two times two is four; four times two is eight; eight times two is sixteen. So when we've multiplied two together four times, we get 16. So 2^4=16.

In general, for any two numbers a and b, a^b means that you multiply a by itself for a total of b factors.

The main point of this section is that

A. x^y, or x raised to the y power, means that you multiply x together for a total of y factors,
or
B. multiplication is repeated addition?

493. There is a special name for exponents of two and three. When a base is raised to the second power, or multiplied by itself, we say that this number is "squared." So 7 x 7 is "7 squared." When a base is raised to the third power, or multiplied by itself for a total of three factors, we say that it is "cubed." So 6 x 6 x 6 is called "6 cubed."

Why are these exponents given these special names? Because if you want to find the area of a square that is for example 5 centimeters on a side, the area is 5 x 5 or "5 squared" or 25 square centimeters. If you want to find the volume of a cube that is for example 5 centimeters on a side, the volume is 5 x 5 x 5 or "5 cubed" or 125 cubic centimeters. So because second powers are often used in finding the areas of squares, they are called squares, and because third powers come in handy for finding the volumes of cubes, they are called cubes.

The purpose of this section was to

A. explain why the volume of a cube is equal to the length of one side multiplied by itself for a total of three factors,
or
B. to explain why second powers are called squares and third powers are called cubes?

494. We can express really big numbers by exponentiation. Which would you rather have, 10 x 10 dollars, or 10^{10} dollars? 10^{10} dollars is a LOT more money. 10 x 10 dollars is 100 dollars. 10^{10} dollars, on the other hand, is $10,000,000,000, or ten billion dollars! How did we get that number? We multiplied 10 together for a total of 10 factors. We remembered our rule that to multiply a whole number by 10, you just add a 0. So we came out with a 1 followed by 10 zeroes.

Here are some other examples of how you usually get bigger numbers by exponentiation than you do by multiplication, when we are talking about positive whole numbers anyway. 5 x 3 is 15, but 5^3 is 5 x 5 x 5 or 125. 3 x 4 is 12, but 3^4 is 3 x 3 x 3 x 3 or 81.

The point of this section is that

A. when x and y are positive whole numbers, x^y usually is bigger, and often lots bigger, than x times y,
or
B. any number to the first power is simply that number itself?

495.When we say that for positive whole numbers, x^y is "usually" bigger than x times y, what are the exceptions to this rule? One exception is when the exponent is 1. Any number raised to the first power is defined as just the number itself. So 2^1 is just 2, and that happens to be the same as 2 x 1. Another exception is 2^2, which is 2 multiplied by itself for a total of two factors, or 2 x 2, or 4. In this case, 2 to the second power happens to be the same as 2 times 2. But be careful not to think that because of these exceptions to the rule, that exponentiation and multiplication are the same thing.

The point the author is trying to make in this section is that

A. exponentiation and multiplication are not the same thing, and with only certain exceptions, the results are very different,
or
B. a number raised to a negative exponent is the reciprocal of the number raised to a positive exponent?

When you multiply two numbers that are powers of the same base, you add exponents

496. Suppose we are asked to multiply 5^3 by 5^2. If we write this product in repeated multiplication form, we get

5 x 5 x 5 times 5 x 5.

So we multiplied 5 by itself for a total of 3 factors, (that's 5^3) and then we multiplied it by itself for two more factors (that's 5^2). So all together, we multiplied 5 by itself for a total of 5 factors. So the result we get is that

$5^3 \text{ x } 5^2 = 5^5$.

Is it a coincidence that if we add the exponents of the two numbers we multiplied together, we get the exponent for the product? No, it happens every time!

Let's do it again, multiplying y^4 by y^3. (I'm going to use the asterisk to stand for multiplication here, so we won't confuse the "times sign" with the variable x.) y^4 is y * y * y * y, and y^3 is y * y * y. So when we multiply them together, we get

y*y*y*y times y*y*y.

This comes out to y multiplied by itself for a total of 7 factors. So we've figured out that

$y^4 * y^3 = y^7$.

The most general way to express this rule is that

$x^a * x^b = x^{(a+b)}$.

The rule that this section is trying to teach is that

A. When you have a power which is itself raised to a power, you multiply exponents,
or
B. When you multiply two numbers that are the same base raised to powers, you add the exponents to get the answer?

When you divide two numbers that have the same base, you subtract exponents.

497. If we add exponents to multiply two numbers that have the same base, what do we do when we divide such numbers? Let's look at x^5 divided by x^3. If we write that as a fraction, using repeated multiplication instead of our exponents, we get this:

$$\frac{x*x*x*x*x}{x*x*x}$$

Now we can "cancel," by dividing top and bottom by the same factors. If we mark out three x's on the top, and three x's on the bottom, we wind up with just

$x*x$, or x^2. So we've found that x^5 divided by x^3 is x^2! It's not a coincidence that 5-3=2, or that we subtracted the exponents, because we took away one x in the numerator for each x in the denominator. If you do this with some other similar division problems, you can convince yourself, if you're not already convinced, that when you divide two numbers that have the same base, you subtract exponents!

To write this as a rule using variables, we can say that $x^a/x^b = x^{(a-b)}$.

The point of this section was to

A. show what happens when you raise a power to a power, such as $(2^2)^3$,
or
B. to explain the rule that when you divide two numbers that have the same base, you subtract the exponents?

Exponents are useful for compound interest.

498. What are exponents good for? They are useful in many, many ways. Let's look at one way: calculating compound interest.

Suppose that you invest $1000 in a bank that pays 5% interest each year. At the end of each year, this bank adds the interest to the principal, or the total amount you have in the bank account. So each year, you earn interest on a little more money than you did the previous year. This is called compound interest, and in this situation we would say that your 5% interest is compounded annually.

So what if you just leave that account alone, and let the 5% interest compound each year, for 3 years? How much will you have at the end of the third year?

The slow way to do a problem like this is to go year by year. OK, at the end of the first year, we get 5% of $1000, which is $50. Now we add this to the $1000, so we have a principal of $1050. At the end of the second year, we get interest of 5% of $1050, which comes out to $52.50. We add this to $1050, to get a new principal of 1102.50. At the end of the third year, we earn 5% interest on $1102.50, or $55.13. We add this to our $1102.50, to get a grand total of $1157.63. That was a lot of work, wasn't it? It also didn't obviously involve exponents. With exponents, we can figure out a much quicker and easier way!

The point of this section was to

A. show how to compute compound interest in the long and slow way, without exponents,
or
B. to show how to use exponents to compute compound interest quickly and easily?

499. Suppose someone were to ask us how much money would be in our account after 50 years? How tedious it would be to sit for a very long time multiplying and adding as we did in the previous section, for a total of 50 repetitions of multiplying and adding.

Let's look at a quicker and easier way to compute compound interest, and first do it with the same problem as we solved in the last section: what's the accumulated bank balance after 3 years.

The first thing to realize is that instead of multiplying by .05 to get the interest, and then adding the interest to the principal, we can just multiply the principal of one year by 1.05 to get the principal of the next year. Why is this? Because each year we take 100% of the principal we had, and increase it by 5%, so that we now have 105% of what we had before. To figure out what 105% of any number is, we convert 105% to a decimal and multiply. In other words, the effect of the interest for any year is to multiply the principal of the previous year by 1.05.

So if we start with $1000, then at the end of the first year, we now have 1000 * 1.05. At the end of the second year, we have 1.05 times as much, or 1000 * 1.05 * 1.05. And at the end of the third year, we multiply our principal by 1.05 again, so that we have 1000 * 1.05 * 1.05 * 1.05.

Instead of writing 1.05 * 1.05 * 1.05, we can use exponents to write our answer as $1000 * 1.05^3$. With a calculator or computer, we can compute this with only a few keystrokes, and we get the same number we got before: $1157.65!

This section demonstrates that

A. when we have $(ab)^n$, it's the same as a^nb^n,
or
B. we can use exponents to compute compound interest easily?

500. Now let's return to our question, how much money would we accumulate if we got 5% interest on $1000, compounded annually, for 50 years? Instead of multiplying 1000 by 1.05^3, we'd multiply 1000 by 1.05^{50}. A few keystrokes on a calculator or computer tells us the answer: $11,467.40. We now have more than ten times as much as we started with! Furthermore, we are now making over $500 a year in interest!

The point of this section is to

A. show that the calculation of compound interest is easy when you use exponents, and also that the amount of money you accumulate through compound interest is surprisingly high,
or
B. to explain exactly which keystrokes on a computer or calculator you use, to raise a number to a power?

The meaning of negative exponents

501. What would we mean by a negative exponent, like 5^{-2} or 10^{-3}? Our original definition of exponents doesn't hold up too well in answering such a question, because we can't think about 5 multiplied by itself –2 times, or 10 multiplied together for a total of –3 factors. Let's answer the question, and then explain why it makes sense. 5^{-2} is defined to be $1/5^2$, or 1/25. 10^{-3} is defined to be $1/10^3$, or 1/1000, or .001. In general, a number raised to a negative exponent is the reciprocal of the number raised to the positive exponent.

Which formula expresses the main idea of the section you just read?

A. A=lwh,
or
B. $x^{-a}=1/x^a$?

502. Why do we define negative exponents to be the reciprocals of the bases raised to positive exponents? Let's think back to our rule of subtracting exponents when dividing. Remember that $x^5/x^3=x^2$? We subtract exponents when we divide numbers that are powers of the same base. What if we were to divide x^3 by x^5? We would subtract the exponents, and get x^{-2}. On the other hand, if we express this as

$\frac{x*x*x}{x*x*x*x*x}$ and cancel three x's from numerator and denominator, we get

$\frac{1}{x*x}$ or $1/x^2$.

So, since we get x^{-2} when we do the problem one way, and $1/x^2$ when we do it a second way, it's good that we have defined x^{-2} and $1/x^2$ to be the same thing!

This section gives an example that illustrates why it's good that

A. we define negative exponents as the reciprocals of the bases raised to the positive exponents, or in other words we define x^{-b} and $1/x^b$,
or
B. we define numbers raised to the one half power as the square roots of those numbers?

The meaning of exponents of 1 and 0

503. A number raised to the first power is just the number itself. For example, 5^1 is just 5. Any number (except 0) raised to the 0^{th} power is an interesting result: 1. Thus 5^0 is 1, 10^0 is 1, and -86^0 is one.

Let's see why these definitions make sense. We'll do just the same thing that we did to see why the definitions of negative exponents make sense.

Let's look at the number x^2/x. If we subtract exponents, we get x^1. If we divide numerator and denominator of this fraction by x, we get x/1, or just x. So it makes sense that x^1 is defined as x!

Now let's look at the division of x^2/x^2. If we subtract exponents, we get x^0. If we divide top and bottom of this fraction by x^2, we get 1/1, or 1! Thus it makes sense that any number – except 0 – to the 0 power is defined as 1.

Why can't we use the same reasoning with 0^0? Because as soon as we set something up like the fraction $0^n/0^n$, getting ready to subtract exponents and get 0^0, we realize that our fraction involves division by 0 and is not defined. Thus 0 to the 0 power is not defined, just as anything else involving division by 0 is not defined.

A summary of this section is that

A. division by zero is not permitted,

or
B. any number raised to the first power is that number itself; any number other than 0 raised to the 0th power is 1.

Scientific notation

504. Here's another of the many things exponents are good for: scientific notation. Scientific notation allows us to express very big or very small numbers in a convenient way. It also allows us to multiply and divide those numbers more easily than when using the standard form of numbers.

Light travels at a speed of about 300,000 kilometers per second. How many kilometers are in a light-year? In other words, how many kilometers does light travel in a year?

To find that out, I take my calculator and multiply 300,000 km/sec by 3,600 sec/hour, and then multiply the answer by 24 hours per day, and multiply the answer by 365.25 days per year! My calculator reads 9.467E12. This means 9.467×10^{12}. This answer is given in scientific notation. It's much less cumbersome than saying that one light year equals about 9,467,000,000,000 kilometers.

To express a number in scientific notation, you have two parts, multiplied together. The first is a number between 1 and 10. The second is a power (negative or positive) of ten. Here are some examples of numbers in scientific notation:

.005 becomes 5×10^{-3},
216,000 becomes 2.16×10^{5},
733 becomes 7.33×10^{2},
and .9173 becomes 9.173×10^{-1}.

Which of the following two numbers is in scientific notation?

A. 4,000,000
or
B. 4×10^{6}?

505. Let's think a little about negative exponents in scientific notation. Why does .003, when converted to scientific notation, become 3×10^{-3}?

First let's ask, what do we have to multiply .003 by to get 3, our required number that is in the range from 1 to 10? We have to shift the decimal point 3 places to the right. That's the same as multiplying .003 by 1000. Now if we want the value to stay the same, we have to divide our number by 1000. (If we multiply by 1000 and divide by 1000, we get the same number we started out with, although perhaps in a different form.) Dividing by 1000 is the same as multiplying by 1/1000, or multiplying by 10^{-3}.

It isn't a coincidence that the negative exponent for the 10 in scientific notation was exactly the number of decimal places we moved to create the number between 1 and 10. It works out that way every time. Just count the number of decimal places you move to the right, and use that as your negative exponent in scientific notation. Similarly, count the number of decimal places you move to the left, and use that as your positive exponent. Here are a couple of examples:

Let's change .000716 to scientific notation. To get a number between 1 and 10, we move the decimal point 4 places to the right, and use –4 as our exponent for 10. So we get 7.16 x 10^{-4}.

This is just the flip side of what we do when we convert a large number to scientific notation. Let's change 93,400 to scientific notation. This number, like all integers, could be written with a decimal point just to the right of it, like this:

934,000.

To get a number between 1 and 10, we move the decimal point 5 places to the left. This is the same as dividing by 10^5. To keep our number the same, we have to multiply by 10^5. So we get 9.34 x 10^5.

When you move the decimal point to the left, you use a positive exponent, and when you move the decimal point to the right, you use a negative exponent. But you don't have to memorize that; you can figure it out.

The major purpose of this section was to

A. help in understanding the relation between how many places you move the decimal point, in which direction, and what your exponent is when you convert a number to scientific notation,

or

B. to understand inverse functions, such as multiplying by 1000 and dividing by 1000?

506. How do you convert numbers back from scientific notation to standard notation? When 10 has a positive exponent, you just multiply by moving the decimal point to the right, for a number of jumps equal to the value of the exponent. When 10 has a negative exponent, you move the decimal point the corresponding number of jumps to the left.

Here are a couple of examples. Suppose we want to convert 8.73 x 10^4 from scientific notation to standard notation. We simply move the decimal point 4 places to the right, and get 87,300.

Suppose we want to convert 6.31 x 10^{-3} to standard notation. We move the decimal point three places to the left, and get .00631.

In other words, to convert from scientific notation back to standard notation, we just multiply the number that's between one and ten by whatever power of ten we have. We follow our rule that multiplying by 10^n means moving the decimal point n places to the right, and multiplying by 10^{-n} is the same as dividing by 10^n, so it means moving the decimal point n places to the left.

This section gave examples and guidelines for

A. changing from standard notation to scientific notation,
or
B. changing from scientific notation to standard notation?

507. How does scientific notation make it easier to multiply and divide big or small numbers? You get to take advantage of the rules of exponents, with the powers of 10. You get to add exponents rather than write a bunch of zeroes. For example, let's say we want to multiply 300,000 by .0002. We could set up this multiplication problem like this:

$$\begin{array}{r} 300000 \\ \text{x}\ \ \underline{.0002} \end{array}$$

and follow our usual rules for multiplying decimals. It's easier, though, to put our two numbers in scientific notation form: $(3 * 10^{5)} * (2 * 10^{-4)}$. Using the associative and commutative laws of multiplication, we change this to $(3*2)(10^5*10^{-4})$. When we multiply 3*2 we get 6, and when we add exponents for the powers of 10 we get 10^1, or 10. So the answer is 6 * 10, or 60.

This section gave an example of how when you multiply two numbers in scientific notation,

A. you add exponents for the powers of 10,
or
B. you use the “E” notation for scientific numbers on a calculator?

A third law of exponents

508. So far we’ve mentioned two laws of exponents. The first is that when you multiply two powers that have the same base, you add the exponents. The second is that when you divide two powers that have the same base, you subtract the exponents. The third law, that we will discuss now, has to do with “powers of powers.” For example, what do we get when we take a^2 and raise it to the third power? In other words, what is $(a^2)^3$? We can figure this out pretty easily without even knowing the third law of exponents. Let’s write $(a^2)^3$ as $a^2 * a^2 * a^2$. We can do this, of course, because any number raised to the third power is just that number multiplied together three times.

Now, to figure out what $a^2 * a^2 * a^2$ is, we just use the first law of exponents. We add exponents when we multiply powers that have the same base. So we add 2+2+2, and we get for our answer, a^6.

Do you notice that the exponent for our answer is the product of the two exponents in our original problem? That is, 6=2*3. This is no coincidence, because we wrote down a^2 three times and added up the exponents, which is the same as multiplying 2*3.

It works out the same way no matter what the exponents are and no matter what the base is. So our third law of exponents is that when you want to take a power of a power, you multiply the exponents. To put this in symbols,

$(a^b)^c = a^{bc}$.

The point of this section was that

A. to multiply powers with the same base, you add exponents,
or
B. to raise a power to a power, you multiply exponents?

A fourth law of exponents

509. Do you remember our distributive law? One way of saying this is that $(a+b)c = ac+bc$. In other words, multiplication is "distributive" over addition. When we want to multiply times a sum, we multiply by each of the terms of the sum.

Our fourth law of exponents tells us that exponentiation is distributive over multiplication. It tells us that if we want to raise a product to a power, we raise each of the factors of that product to the power and multiply them together.

This law is a lot simpler in symbols than it is in words. The law is:

$(ab)^n=a^nb^n$.

Let's check it out with some simple numbers. Does $(2*3)^2=2^2*3^2$? The left side is six squared, or 36. The right side is 4 times 9, or 36. So it works.

The main point of this section is that

A. $(a^n)^m=a^{nm}$,
or
B. $(ab)^n=a^nb^n$?

510. Let's see if we can explain why this law works. Let's start with raising (ab) to the second power, or squaring it. By definition,

$(ab)^2 = (ab)(ab)$.

Now all we have to do is to use the commutative and associative laws of multiplication to rearrange the factors on the right side. We get

$(ab)^2 = (a*a)(b*b)$.

Putting the right side in exponent notation gives us

$(ab)^2 = a^2b^2$.

And this is what we were trying to prove. If we did the same thing with raising (ab) to the third or fourth or any other power, we would proceed in exactly the same way.

In order to prove our fourth law of exponents, we

A. used a very complicated theorem of higher mathematics,
or
B. just used the definition of raising a number to a power plus the commutative and associative laws of multiplication?

A fifth law of exponents

511. The fifth law of exponents says that a fraction raised to a power is equal to the numerator raised to that power, over the denominator raised to that power. In symbols, $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.

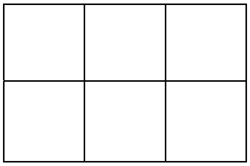
It’s easy to see why this works if you’re raising a/b to the second power, for example. By definition, a/b to the second power is a/b * a/b. Now, if we multiply numerator times numerator, and denominator times denominator, as we do when we multiply any two fractions, we get a^2/b^2. And the same process would work if we were raising a/b to the third, fourth, or any other power.

The purpose of this section was to

A. state and explain the law of exponents that tells how to raise a fraction to a power,
or
B. state and explain the law of exponents that tells how to divide powers of the same base?

Chapter 26: Finding Areas of Parallelograms, Triangles, Trapezoids, and Circles

512. We talked earlier about how to find the area of a rectangle. Let's review a bit. We defined area as how many little squares of a certain size (say one that's a centimeter on each side, or a square centimeter) could fit inside another figure. For a rectangle that is three units long and two units high, for example, like the one below,



we multiply three by two to find that there are six square units of area inside the rectangle. This is logical because there are two rows and three columns of little squares.

Thus the formula for the area of a rectangle is A=lw, where A stands for area, w stands for width, and lw means the length multiplied by the width.

The basic formula for the area of a rectangle is the one from which every other area formula is derived! We will find that this is true even for the area of a circle. When you study calculus, you will find that the basic formula for the area of a rectangle is a central building block involved in finding areas of complex and curvy shapes.

The purpose of this section was to

A. explain how to find the area of new figures,
or
B. review how to find the area of a rectangle, and to say why this is important?

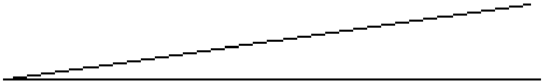
513. Before starting, let's talk about some of the meanings of the words we use in describing geometric figures. First, what's the difference between a line, a ray, and a line segment? Lines are imagined as going infinitely far in both directions –

never ending. Rays are imagined as starting at a point and going forever in one direction. Line segments are pieces of lines, with starting points and end points. In the real world, you always see line segments and not lines or rays, unless you can see forever! The things that make up the sides of rectangles, squares, triangles, and other such figures are line segments, not lines or rays.
This section pointed out that the imaginary long thin things that go forever in both directions are called

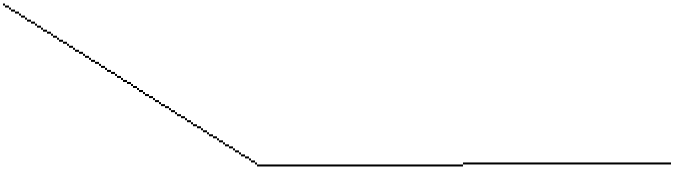
A. lines,
or
B. line segments?

514. When lines, rays, or line segments cross each other or start at the same point, they form angles. The bigger the opening between two rays or line segments, the bigger the angle.

Here's a fairly little angle. In degree measurement, it's around 10 degrees.



And here's a much bigger one, which measures somewhere around 150 degrees:

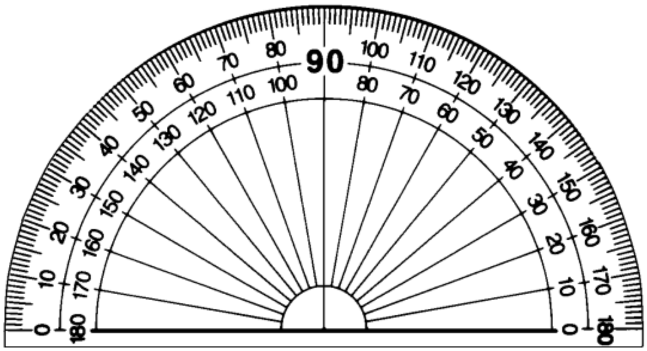


The purpose of this section was to

A. illustrate what angles are,
or
B. to define radians as units with which you measure angles, that have certain advantages over degrees?

Reading about Math

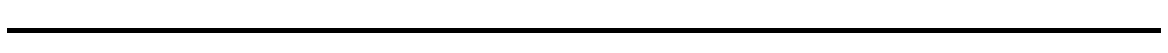
515. We measure angles with an instrument called a protractor. You put the middle of the straight part of the protractor right at the vertex of the line segments, which is where the two line segments meet. You match up the line on the protractor that points to "0" with one of the two line segments. Then you find where the other line segment falls on the protractor and read off the angle. Protractors are usually made of clear plastic, so that you can put it over an angle and see both the numbers on the protractor and the angle you're trying to measure. Here's a picture of a protractor.



The main purpose of this section was to

A. explain how angles are measured,
or
B. explain why a protractor is usually made of clear plastic?

516. What if you had two line segments come together to make a straight line segment? Here's an example.



Imagine that the vertex is right in the middle, even though you can't tell exactly where the vertex is, and it doesn't make any difference, when you're measuring the angle. If you put the protractor on the vertex of this angle, and lined one of the two segments with the direction of 0 degrees on the protractor, where would the other line segment fall? The answer is 180 degrees. It's an important fact that when two line segments come together to make a straight line segment, the angle between them is 180 degrees; 180 degrees is called a straight angle. Also: if you have several angles that combine so that two line segments make a straight line segment, the separate angles add up to 180 degrees.

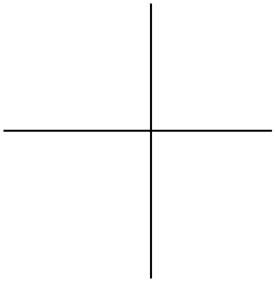
A summary of this section is that

A. when line segments come together at an angle that makes a straight line segment, their angle is 180 degrees,
or
B. the angles of a triangle add up to 180 degrees?

517. What are perpendicular line segments? Perpendicular lines form right angles, or 90 degree angles. These two line segments are perpendicular to each other, and the angle is 90 degrees:

As are these:

And these are also perpendicular to each other:

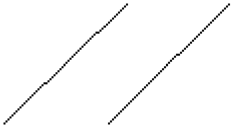


When two line segments cross each other perpendicularly to each other, they form four equal angles.

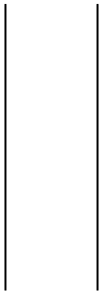
The purpose of this section was to

A. review what perpendicular lines or line segments are,
or
B. to prove that perpendicular line segments crossing each other create four equal angles?

518. Parallel lines don’t cross each other, because they slope in exactly the same way. Here are a pair of parallel line segments:



And below are two more parallel line segments:



And here are two more:

If two lines are parallel, they

A. get closer together and farther apart,
or
B. always stay the same distance apart?

Angles of a triangle

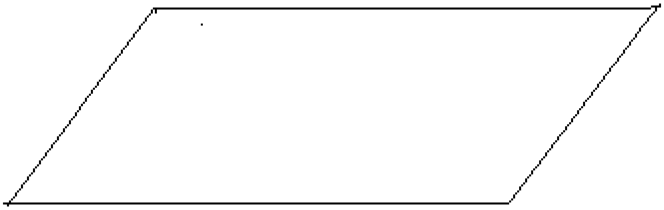
519. Before we stop talking about angles, we should mention a very important fact: the angles of any triangle add up to exactly 180 degrees. So if we know any two angles of a triangle, we can always find the third. We do this by adding the two together, and subtracting our sum from 180 degrees. Here's an example: two angles of a triangle are 60 degrees and 40 degrees. What's the third angle? When we add 60 and 40, we get 100. So we know that 100 plus the third angle has to equal 180. So the third angle equals 180 minus 100, or 80. Let's check: 60+40+80 does equal 180.

A major point this section makes is that

A. if you are given two angles of a triangle, you can always find the third (and you do it by adding the two angles and subtracting the sum from 180),
or
B. if you cut out a triangle from paper, and fold the top vertex down and the side vertices inward, you'll see that the three angles fit together to make a straight line, or a 180 degree angle?

Areas of parallelograms

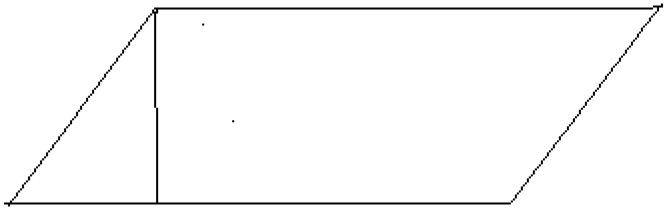
520. Let's think about finding the area of a parallelogram. A parallelogram looks like you'd taken a rectangle with hinges, and moved the top and sides. A parallelogram has its opposite sides equal and parallel. Here's what a parallelogram looks like:



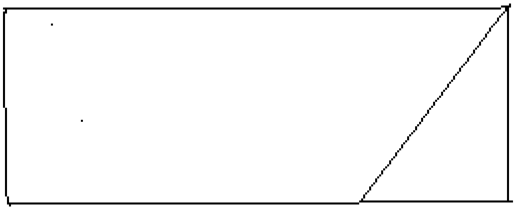
The purpose of this section was to

A. explain what a parallelogram is,
or
B. explain how to find the area of a parallelogram?

521. Let's draw a line from the upper left corner of this parallelogram down to the bottom line of it. We'll draw our line perpendicular to the bottom line. The line we draw is called the height of the parallelogram, and the bottom line of the parallelogram is called the base.



Do you see that triangle over on the left side of the parallelogram? Let’s cut it off and slide it to the right, so that it fits right next to the right side of the parallelogram. Here’s what we get:



This is a figure that we recognize as a rectangle. What are its dimensions? The base is the same as the base of the original parallelogram, because all we did to that base was to cut off a line segment on one end and put it back on the other end. The width of the rectangle is just the height of the original parallelogram. And since we haven’t shrunk any parts of the figure, the area of the rectangle is the same as that of the original parallelogram. So: if the area of the rectangle is its length times its width, the area of the parallelogram is its base times its height!

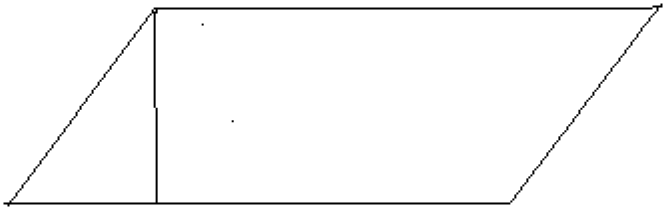
Put in symbols, A=bh, where A is the area of a parallelogram, b is the length of its base, and h is its height.

The main conclusion of this section is that

A. you can cut off a triangle from a parallelogram, slide it, and make a rectangle, or
B. the area of a parallelogram is its base times its height?

522. When you are finding the area of a parallelogram, don’t forget what the height is. The height is the length of a line drawn perpendicular to the base, to the

other side. It's not one of the sides of the parallelogram (except in the special case where the parallelogram is also a rectangle).



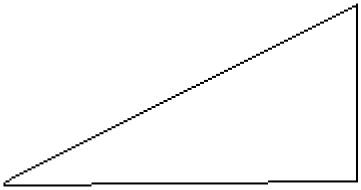
So in the drawing above, the height of the parallelogram is the length of the line going straight up and down, not that of the ones slanting from lower left to upper right.

The author's purpose in this section is probably

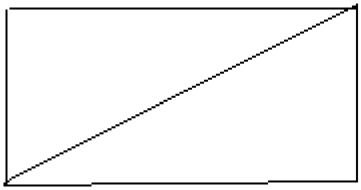
A. to help you keep from making an error that people often make,
or
B. to explain to you what the word "perpendicular" means?

Areas of triangles

523. Now let's think about the area of a triangle. Below is a special type of triangle, a right triangle, where two of the sides are perpendicular to each other.



We call the bottom line of this triangle the base, and the line going straight up from it, the height. Now let's take an exact copy of this triangle, turn it upside down, and put it next to the triangle we've already got. We get something that looks like this:

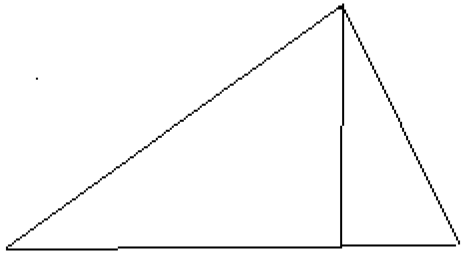


And we recognize that figure as a rectangle! The length of that rectangle is the base of the triangle, and the width of it is the height of the triangle. So the area of the rectangle is the base times the height. And since the original triangle is half of the rectangle, the area of the triangle is one-half the base times the height! To put this in a formula, A=1/2 bh, where A is the area of the triangle, b is the length of the base, and h is the length of the height.

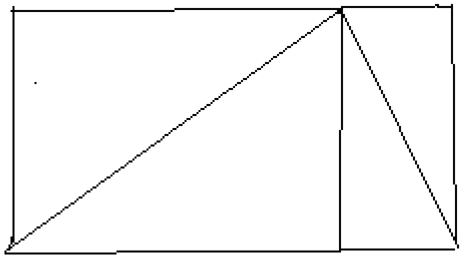
The main point of this section is

A. to explain why the area of a right triangle is 1/2 base * height,
or
B. to show that if you put two identical right triangles together, you can make a rectangle?

524. We've shown why the area of a right triangle is 1/2 base * height. But what about other triangles? The same formula still holds! Below I've drawn a triangle, with a perpendicular line straight up from the base to the vertex (or corner) of the triangle just opposite that base. That perpendicular line to the vertex is called the height of the triangle.



The line for the height of the triangle (which is also called the altitude of the triangle) divides the triangle into two other triangles, doesn't it? And each of those is a right triangle. So now we can do the same thing we did with the right triangle we used before. We can make an exact copy of each of those two right triangles, and make rectangles out of them. When we have placed the two duplicate triangles correctly, we've made a big rectangle, whose area is twice that of the original triangle. The length of this rectangle is the same as the base of the triangle, and the width of this triangle is the height of the triangle. So one half the area of this rectangle, or 1/2 base * height, is the area of the original triangle!



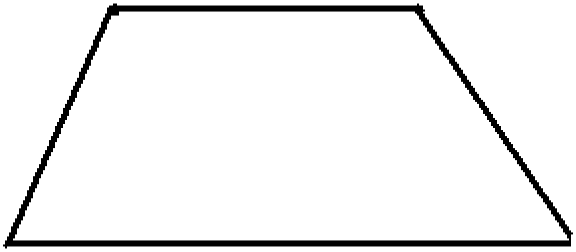
The purpose of this section was to

A. show that the area of a right triangle is 1/2 base * height,
or

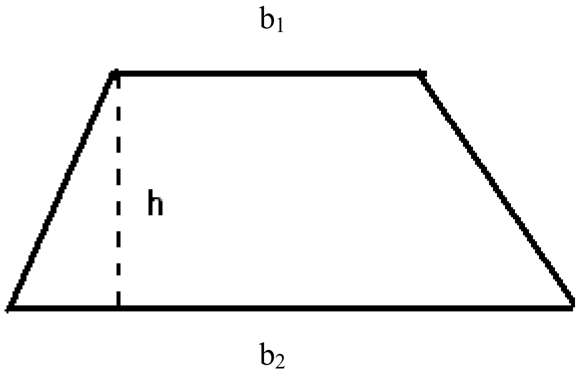
B. to show that the areas of triangles other than right triangles are also 1/2 base * height?

Areas of trapezoids

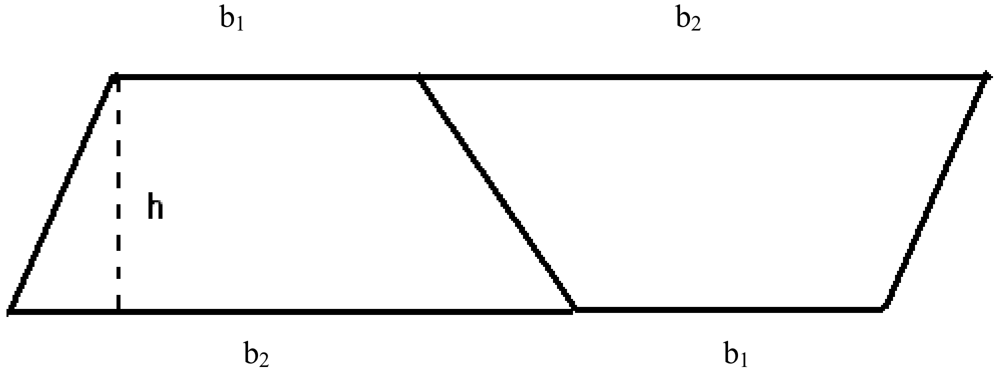
525. Now let's think about the areas of trapezoids. A trapezoid is a four-sided figure where two sides are parallel and two sides are not parallel. Here's one possible way that a trapezoid can look:



We will call the lengths of the parallel sides of the trapezoid "b_1" and "b_2." (Pronounced b sub-one and b sub-two.)
We can draw a height or altitude for the trapezoid, by making a perpendicular line between the parallel sides, like this:



Now let's make a duplicate of the trapezoid, turn it upside down, and put it right next to our original trapezoid. Here's what we get:



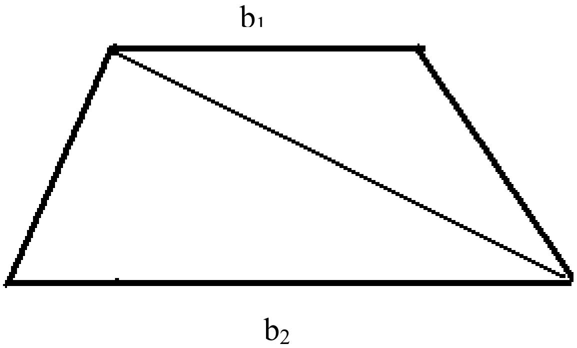
What kind of figure is this? It's a parallelogram! The base of the parallelogram is b_1+b_2 units long, and the height of the parallelogram is the height of the original trapezoid. The area of that parallelogram is $h(b_1+b_2)$, by our $A=bh$ formula. Since the parallelogram is twice the area of the original trapezoid, the original trapezoid is 1/2 the area of the parallelogram we made. So the area of the trapezoid is

$A=1/2\ h(b_1+b_2)$, where h is the height of the trapezoid and b_1 and b_2 are the lengths of the parallel sides.

The strategy we used to find the formula for the area of a trapezoid was to

A. make a duplicate trapezoid, turn it upside down, and attach it to make a parallelogram; find the area of the parallelogram by $A=bh$, and take one-half of that for the area of the trapezoid;
or
B. cut the trapezoid into two triangles and compute the area of each of those triangles, then add them together?

526. One of the interesting things about math is that there are often many different ways of proving things. Let's look at a different way to prove the formula for the area of a trapezoid. Let's draw a line from the upper left corner of a trapezoid to the lower right. This splits the trapezoid into two triangles, like this:



The area of the triangle on the lower left is 1/2 b_2*h. The area of the triangle on the top right is 1/2 b_1*h. In both cases the height of the triangles equals the height of the trapezoid. The area of the trapezoid is the sum of the areas of those two triangles, or

$A= 1/2\ h*b_1 + 1/2h*b_2$.

Do you remember our distributive law? We can use it to rewrite the formula as

$A= 1/2\ h\ (b_1+b_2)$,

which is the same formula we got the other way!

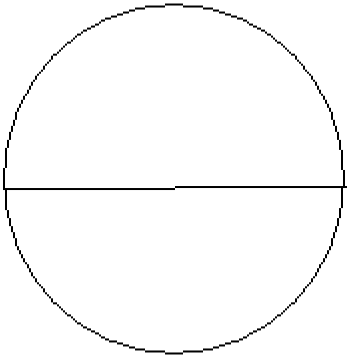
This section demonstrated that

A. the area of a triangle is 1/2 bh,
or
B. there are at least two ways to show that the area of a trapezoid is 1/2 $h(b_1+b_2)$; the one described in this section splits the trapezoid into two triangles.

Areas of circles

527. Now let's figure out how to find the area of a circle. But before doing that, we will need to think about the number π, or pi. Pi is equal to approximately 3.14. For this reason, some people celebrate "pi day" on March 14 (3/14). Because pi is more closely equal to 3.14159265, people sometimes make up "pi sentences" like this: "How I like a sweet milkshake on summer noons!" Why is this a pi sentence? Because the number of letters in each word goes in the same order as pi's digits!

Why is pi so important? It tells us how many times greater the circumference of any circle is than the diameter. The circumference is the distance all the way around the circle. The diameter is the length of a line going from one side of the circle to the other, straight through the center. Here's a picture of a circle with a diameter drawn:



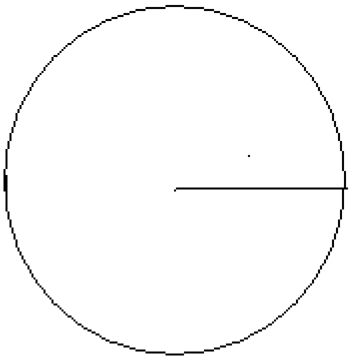
As you can see, the diameter splits the circle into two equal parts.

If you get a tape measure and very carefully measure the distance around this circle, and the length of its diameter, you should find that the circumference is about 3.14 times as long as the diameter. You can do this with other circular objects, like plates and pots. The ratio comes out the same, except for the errors we make in trying to measure accurately, or the error that comes if the object isn't exactly circular.

The main point this section has made is that

A. the area of a circle is πr^2,
or
B. the number π, or pi, is about 3.14, and it is how many times bigger the circumference of a circle is than the diameter?

528. The radius of a circle is one-half the diameter. The radius goes from the center of the circle to a point on the circle. Here's what a radius looks like:



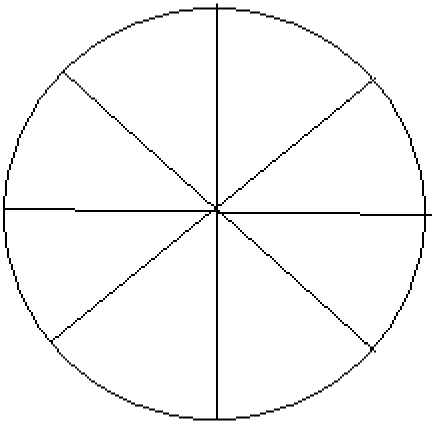
So if the circumference is pi times bigger than the diameter, and the diameter is twice as big as the radius, how many times bigger is the circumference than the radius? The answer is 2 pi. The circumference is 2 pi times the radius, or

$C=2\pi r$

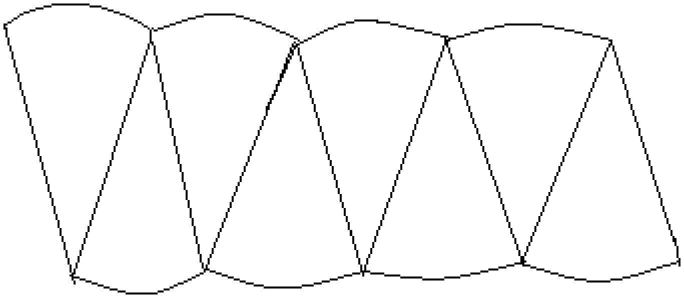
A direct consequence of this section is that

A. the circumference of a circle is about 6.28 times as long as the radius (since 6.28=2*3.14),
or
B. the area of a circle is proportional to the square of the radius?

529. Now we're ready to figure out the area of a circle. Let's start by taking the circle and slicing it into eight slices, just as people often slice up a circular pizza. Here's how the circle looks when it's been sliced:



Now let’s take these pieces out and rearrange them. (The next time you get a pizza, you might try this.) You alternate, putting one wedge with the curved side up and the next one with the curved side down. You get something that looks like this:

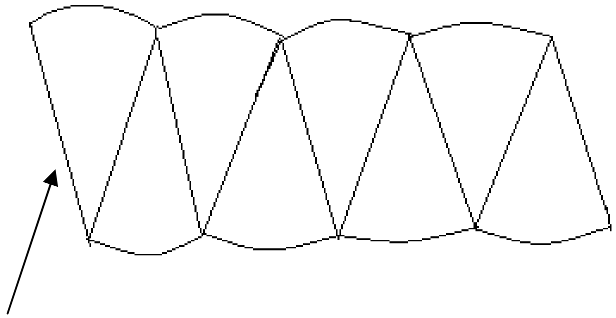


This shape looks roughly rectangular, doesn’t it? Now please imagine what it would look like if we cut the circle into more and more slices that were smaller and smaller. The top and bottom of our figure wouldn’t be so curvy, would they? In fact, if you cut the circle into let’s say 1000 equal wedges, the figure that you would make if you put them together in the same way would be almost exactly a rectangle!

The point of this section is that

A. you can cut a circle up and rearrange it, and if you cut it into enough small pieces, the rearrangment comes very close to being rectangular,
or
B. it would be very difficult to actually cut a circle into 1000 equal pieces?

530. We can make the shape from wedges as close as anyone wants to rectangular, by cutting more and more wedges to rearrange. Let’s use our 8-wedge illustration, and think of what the length and width of this “close to rectangular” shape is.



The width of it is the radius of the circle. Why? Because if we look at any given wedge, the two sides of it go from the circle to the center of the circle, and lines that do that are radii. (Radii is the plural of radius.)

What’s the length of this “almost rectangle?” The total length of the curvy part that makes up the top and bottom is the circumference of the circle, because that’s where that part came from: the circumference was cut up into parts which were laid next to each other. Since the whole circumference is $2\pi r$ units long, the curvy top and the curvy bottom of our figure are both half that, or πr units long.

Now we have something like a rectangle, with a length that is πr and a width that is r. Let’s just multiply the length by the width to get the area. What we get is that
$A= \pi r*r$

and since r*r is the same as r^2, we can write the formula for the area of a circle in the usual way:

$A= \pi r^2$,

where A is the area, π is about 3.14, and r is the radius of the circle!

The conclusion of this section is that

A. pi is a number that can't be expressed as one whole number divided by another, and for that reason is called an "irrational number,"
or
B. the area of a circle is equal to pi times the square of the radius of the circle?

531. Let's summarize. In this chapter we've figured out the formulas for the areas of several geometric figures. Here they are:

parallelogram: $A=bh$,

where b is the base and h is the height (height is the perpendicular distance between the base and the side opposite it).

triangle: $A=1/2\ bh$,

where b is the base and h is the height (height is the perpendicular distance from a side of the triangle to the vertex opposite it).

trapezoid: $A=1/2\ h(b1+b2)$,

where b1 and b2 are the lengths of the parallel sides, and h is the perpendicular distance between them.

circle: $A= \pi r^2$,

where π is about 3.14 and r is the length of the radius of the circle.

We’ve also spoken about two other formulas: One is the circumference of a circle in terms of its diameter,

$C= \pi d$,

where C=circumference and d=diameter.

The other is the circumference of a circle in terms of its radius,

$C=2\pi r$,

where C is the length of the circumference and r is length of the radius of the circle.

What’s a fact that was reviewed in this section?

A The area of a triangle is ½ bh, where b is the length of the base, and h is the length of the height.
or
B. The surface area of a cube is $6s^2$, where s is the length of an edge of the cube.

Chapter 27: Roots and Fractional Exponents

Roots are inverse functions of exponents

532. Do you remember that we've talked about inverse functions? Suppose we imagine a function as a machine constructed so that when you put in a number (let's call it the input number) you get out a number (let's call it the output number). The inverse function is another machine that we could use to get our input number back again. We simply put the output number into the inverse function machine, and we get back the number we started with. If our original function is to add 5, the inverse function is to subtract 5. If the original function is to multiply by 9, the inverse function is to divide by 9. If the original function is to take the reciprocal, the inverse function is to take the reciprocal again.

The purpose of this section was to

A. give a definition of the square root function,
or
B. to review what is meant by inverse functions?

533. Let's think about the function $y = x^2$. This is just the "squaring" function. If we apply this function to 3, we get 9. If we apply this function to 5, we get 25. What sort of function would get us from 9 back to 3, and 25 back to 5? In the case of 25, we could make up a long-winded name for that function. We could call it the "number which when multiplied by itself gives 25" function. But since this phrase is a little to cumbersome, people have made a shorter name for it, which is the square root of 25. For any number x, the square root of x is the number which, when squared, gives x.

The square root of 25 is written, in mathematical symbols, like this:

$\sqrt{25}$.

The square root of x is written, in mathematical symbols, like this:

$\sqrt{x}$.

The main idea of this section is that

A. the square root of x is the number which when multiplied by itself gives x,
or
B. 5 multiplied by itself gives 25?

534. There's just a little complication that makes the definition we gave for square roots in the previous section a little bit inexact. Suppose we want a number which when multiplied by itself gives 25. The first number that comes to mind is 5, but there is another. Can you think of what it is? Since negative numbers multiplied by negative numbers give positive numbers, -5*-5 also equals 25.

This presents a little problem, because when we define a function, we define it so that when we put one number in, we get only one number out. So to get around this problem, we define the square root function, and the meaning of the square root symbol, to be just the positive number of the two we could possibly use. Thus,

$\sqrt{25}$ = +5, and not –5.

When we want to speak about both of the numbers which when squared give 25, we write this:

$\pm\sqrt{25}$. The symbol, ± is read as "plus or minus."

This section makes the point that:

A. $\sqrt{x}$ is defined to be a positive number,
or
B. the cube root of x is defined to be the number which when cubed gives you x?

535. Let's suppose that you have an equation to solve: $x^2 = 25$. To solve this we use the golden rule of equations and do the same thing to both sides. We would take the square root of both sides, but in this case, we want to get both the answers that make the original equation true. So we get two solutions to the equation: +5, and –5. We could write this as ± 5.

This section made the point that:

A. equations where the highest power of x is 2 are called quadratic equations,
or
B. equations where x^2 equals a certain number will have two answers, one negative and one positive?

536. Let's just look at some concrete examples of how squaring and taking the square root are inverse functions. If we start with 2 and square it, we get 4. If we then put the 4 into a "square root function machine," we get our original 2 back. If we start with 10 and square it, we get 100. If we take the square root of 100, we get back to 10.

This section

A. gave some examples of how squaring and taking the square root are inverse functions,
or
B. made an important point about squaring negative numbers?

537. For all numbers greater than 1, the square root of the number is less than the number itself. For example, 2 (the square root of 4) is less than 4, 3 is less than 9, 4 is less than 16, and 1.1 is less than 1.21.

For fractions or decimals between 0 and 1, the square root of the number is greater than the number itself. For example, the square root of 1/4 is 1/2, which is greater than 1/4. The square root of 1/9 is 1/3. The square root of .16 is .4.

0 and 1 are both their own square roots, because 0 squared is 0 and 1 squared is 1.

This section talked about

A. how cube roots of negative numbers are negative,
or
B. which numbers have square roots that are less than, greater than, or equal to the numbers themselves?

Rational and irrational numbers

538. When we take the square roots of certain numbers, for example, 1, 4, 9, 16, 25, and so forth, we get integers. Numbers like these, which have integers as square roots, are called perfect squares. When we take the square roots of numbers other than perfect squares, we get an interesting type of number. These numbers have decimal parts that go on forever and never repeat themselves. So for example, the square root of 2 rounded to the eighth decimal place is this: 1.41421356. But this is not exactly the square root of 2; it is an approximation. No matter how many decimal places we use, we can only give an approximation, and never the exact square root of 2. The same thing goes for the square root of 3, 5, 6, 7, and all the other numbers that aren't perfect squares. These nonrepeating decimals that go on forever are called irrational numbers.

Do you remember that we gave, in an earlier chapter, a procedure for changing repeating decimals into fractions? Any number that can be written as a fraction, in other words the quotient of two integers, is called a rational number. Numbers like 1/3, 4/1, and .131313... are all rational numbers.

This section

A. defines rational and irrational numbers, and tells us that the square roots of numbers other than perfect squares are irrational,
or
B. explains how you change a repeating decimal into a fraction?

539. The numbers that you get when you take roots of non-perfect powers are called irrational numbers. The numbers we can express as fractions are rational numbers. (Rational numbers include integers.) We give a name to the set which includes both rational and irrational numbers – this is called the set of "real numbers."

These numbers are "real" because we can conceive of a point on the number line corresponding to each real number. If we could draw a square with area exactly two square centimeters, the length of each side would be exactly the square root of two centimeters.

The point of this section is that

A. integers include zero, the counting numbers, and the opposites of the counting numbers,
or
B. the set of real numbers is all those that could correspond to a point on the number line; the set of real numbers is the union of the set of rational and irrational numbers?

Real numbers and imaginary numbers

540. What do we get when we try to take the square root of a negative number? For example, what is the square root of –4? It isn't –2, because squaring –2 gives us +4, not –4. (This is because a negative times a negative always equals a positive.) And the square root of –4 certainly isn't +2. The answer is that we can't come up with any real number which when squared gives us a negative number.

What's the square root of –1? Again, there is no such thing, when we are thinking of real numbers. However, mathematicians have found it useful to imagine a number which when squared gives –1! They have called this number i. If we square 2i, we would get 2^2*i^2, or 4*-1, or –4. Making up the number i lets us give answers for the square roots of negative numbers, as multiples of i, even though these numbers have no location on our number line. Multiples of i are called *imaginary numbers*.

This section

A. defined imaginary numbers, which are not points on the number line, but multiples of "i," which is the imagined square root of –1?
or
B. made the point that when you are asked for solutions to equations on tests, the test-makers will usually be looking for real solutions, not solutions involving imaginary numbers?

Roots other than square ones

541. There are lots of different roots other than square roots. The cube root of x is defined as the number, which when cubed, results in x. So for example, two is the

cube root of eight, because 2*2*2 = 8. Just as 2 cubed is the same as 2 to the third power, the cube root of 8 is the same as the third root of 8.

The fourth root of x is the number which when raised to the fourth power results in x. For example, the fourth root of 81 is 3, because 3^4, or 3*3*3*3, = 81.

In general, the nth root of x, is the number, which when raised to the nth power, results in x.

The purpose of this section was to

A. show that 3 to the 4th power = 81,
or
B. define what is meant by roots other than square roots?

542. We write the cube root of x, using the same symbol that we used for the square root of x, only with a three in the upper left. It looks like this:

$\sqrt[3]{x}$.

The fourth root of x would be written as follows:

$\sqrt[4]{x}$.

We could really write the square root of x like this:

$\sqrt[2]{x}$,

but we wouldn't need to, because everyone has agreed that our symbol $\sqrt{\ }$, which is called the "radical" sign, with no number, means square root.

This section

A. defined fractional exponents,
or
B. defined the radical sign, and explained how it's used to represent roots other than square roots?

Exponents that are unit fractions

543. Now let's do something important: let's explain the meaning of fractional exponents. What would we mean by something like $x^{1/2}$ or $x^{1/3}$? (We would read these as "x to the one-half," and "x to the one-third.") Let's first think about $x^{1/2}$. Suppose that we were to take $x^{1/2}$ and square it. We could write this as $x^{1/2} * x^{1/2}$. If we write it this way, we can use our first law of exponents and add exponents to get x^1, which is just x. Or, if we preferred, we could have written $(x^{1/2})^2$.This is a power raised to a power, the situation where we use our third law of exponents. Our third law of exponents says that in this situation you multiply the exponents. 1/2 * 2 gives us 1, so our answer again is x^1, or just x.

So what we have found out so far is that if we square $x^{1/2}$, we get x.

The main question that the author is working on in this section is

A. what is the meaning of fractional exponents,
or
B. what is the meaning of negative exponents?

544. In the previous section, we figured out that when we square $x^{1/2}$, we get x. To give a specific example, if we were to square $9^{1/2}$, we would get 9. So $x^{1/2}$ means "the number which when squared gives x." Does this sound familiar? This is the exact definition we gave for the square root of x, isn't it? We have just figured out that $x^{1/2}$ and $\sqrt{x}$ mean exactly the same thing!

The conclusion of this section is that

A. $x^{1/2} = \sqrt{x}$,
or
B. $x^{-2} = 1/x^2$?

545. Just as $x^{1/2}$ is $\sqrt{x}$, $x^{1/3}$ is $\sqrt[3]{x}$. Let's understand why this is the case. We'll use exactly the same reasoning that we used before. Let's take $x^{1/3}$ and raise it to the third power. Multiplying exponents, we get

$(x^{1/3})^3 = x^1 = x.$

This tells us that $x^{1/3}$ is the number which when cubed gives us x. And this, by definition, is the cube root of x.

Using exactly the same reasoning, we can figure out that $x^{1/4}$ is the fourth root of x, $x^{1/5}$ is the fifth root of x, and so forth.

A way of stating the point of this section is that

A. fractions with 1 in the numerator are called unit fractions,
or
B. any number x raised to the 1/n power is the same as the nth root of x?

Exponents that are non-unit fractions

546. Even though the answer to the question for the previous section was B, the statement given in answer A is correct. Fractions like 1/3 and 1/5, with 1 in the numerator, are called unit fractions, where as fractions like 2/3 and 3/5 are non-unit fractions.

So now we have figured out the meaning of exponents that are unit fractions. We have figured out that any number x raised to the 1/n power is just the nth root of x. What about non-unit fractional exponents? What, for example, would be the meaning of 27 to the 2/3 power, or $27^{2/3}$? What's the meaning of any number x to the 2/3 power?

The question the author is starting to work on is

A. what happens when you raise a fraction to a certain power,
or
B. what is the meaning of non-unit fractions as exponents?

547. In figuring out the meaning of non-unit fractional exponents, our third law of exponents comes in handy. This is the law that says that when you raise powers to powers, you multiply the exponents. Let's state this law in symbols, only reversing the order of the equation that we originally gave. Our third law is:

$x^{ab} = (x^a)^b$.

Now let's consider that any non-unit fraction can be thought of as the product of an integer and a unit fraction. For example, 2/3 = 1/3 times 2. 3/5 = 1/5 times 3.

So if we're raising x to the 2/3 power, we can think of it as x to the 1/3 times 2 power. And this, by our third law of exponents, is the square of x to the 1/3 power, or the square of the cube root of x. Here's how this looks in symbols:

$x^{2/3} = x(^{1/3*2}) = (x^{1/3})^2 = \left(\sqrt[3]{x}\right)^2$.

What's an example of the main point of this section?

A. To raise x to the 3/4 power, you take the fourth root of x and then cube it,
or
B. to raise x to the 1/4 power, you simply take the fourth root of x?

548. Let's look at an example or two. What's 27 to the 2/3 power? To get this we first take the cube root of 27, and then square it. The cube root of 27 is 3, because 3*3*3 = 27. When we square 3, we get 9. So 27 raised to the 2/3 power is 9.

What is 10,000 raised to the 3/4 power? It's the fourth root of 10,000, cubed. The fourth root of 10,000 is 10, because 10^4 = 10,000. If we cube 10, we get 1,000. So 10,000 raised to the 3/4 power is 1,000.

The point of this section was to

A. explain the third law of exponents,
or
B. give some examples of how to raise a number to a power that is a non-unit fraction?

A root of a product is the product of the roots

549. Do you remember the rule that we called the fourth law of exponents? It was that

$(ab)^n = a^n b^n$.

This relationship between exponentiation and multiplication reminded us of the distributive law, which tells a relationship between addition and multiplication: $(a+b)c = ab+ac$.

We've just talked about how square roots, cube roots, and so forth, are really exponents – fractional exponents. Do they obey the fourth law of exponents? For example, does

$(ab)^{1/2} = a^{1/2} b^{1/2}$?

This is the same as asking, does

$\sqrt{ab} = \sqrt{a} * \sqrt{b}$?

The question this section raises is

A. does the square root of a product equal the product of the square roots?
or
B. is the fourth law of exponents more important than the third?

550. The answer to the question we raised in the previous section is yes, with one exception. In other words, the fourth law of exponents usually holds for fractional exponents, which are roots, just as it does for exponents that are integers. Let's check out this rule with some concrete numbers and make sure it works. Is it true that

$\sqrt{9*4} = \sqrt{9} * \sqrt{4}$?

On the left side of this equation, we get the square root of 36, which is 6. On the right side, we get 3*2, which is also 6. So it comes out right. And it also works for any other concrete numbers that you might choose.

The main point of this section is that

A. The square root of 9 is 3, because 3 squared is 9, and the square root of 4 is 2, because 2 squared is 4?
or
B. The square root of any product is equal to the product of the square roots?

551. The rule for square roots that we have just stated gives us a way of simplifying square roots of numbers that are multiples of perfect squares. The word “simplifying,” in this case, means making the number under the square root sign as small as we can get it. For example, let’s think about the square root of 12. 12 has a factor which is a perfect square, namely 4. We can reason as follows:

$$\sqrt{12} = \sqrt{4*3}$$

which, according to the rule we just stated,

$$= \sqrt{4} * \sqrt{3}$$

and since the square root of 4 is 2,

$$= 2\sqrt{3}\,.$$

So we’ve simplified the square root of 12, and have shown that it equals twice the square root of 3.

This section

A. showed how to make decimal approximations for square roots,
or
B. showed how to simplify square roots of numbers that are multiples of perfect squares?

552. You may remember that I said that there is one type of exception to the law we have just stated, which is that a root of a product is equal to the product of the roots. Most elementary texts will not even tell you about this exception, so you can skip this section if you want. The exception comes when you are taking an even root of a negative number. What is the square root of –4? The answer isn’t –

2, because –2 * -2 gives 4, not –4. And the answer certainly isn't 2. Mathematicians have created something called imaginary numbers, so they can work with these sorts of problems. Thus they can say that the square root of –4 is 2i, where i is the square root of –1, and i squared is –1. It's only when you're dealing with imaginary numbers that the law we just stated above doesn't work. When you use imaginary numbers correctly, you find that the square root of –9 times the square root of –4 turns out not to be the square root of –9 * -4 or 36. It is 3i times 2i, or $6i^2$, which is not 6, but –6. The product of the square roots (or -6) is not the square root of the product (which is 6)!

But when you're dealing with real numbers, the sort that you can locate on a number line, the law about the product of roots equaling the root of the product is true.

In this section the author's goal was to

A. fully explain imaginary numbers and how you use them,
or
B. tell about the exception to the rule about the product of roots, without going too far into imaginary numbers?

A root of a quotient is the quotient of the roots

553. There is a law of square roots that corresponds to our fifth law of exponents. The fifth law of exponents says that

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} .$$

The corresponding law of square roots is that

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} .$$

Let's check it out with some real numbers. Does

$$\sqrt{\frac{100}{4}} = \frac{\sqrt{100}}{\sqrt{4}} \text{ ?}$$

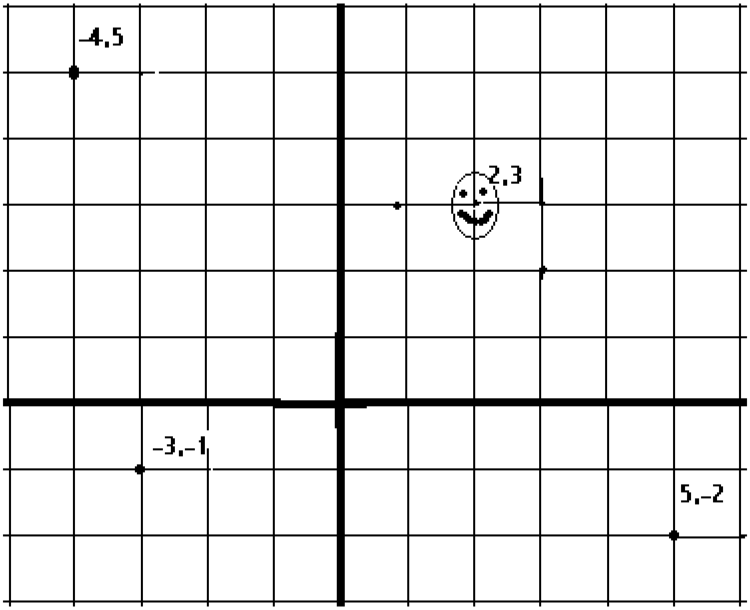
It does work out, because on the left side we get the square root of 25, which is 5. On the right side, we get 10 divided by 2, which is also 5. It works for any real numbers that we can choose (except that 0 can’t be in the denominator).

(By the way, we also get exceptions to this rule when we are working with the even roots of negative numbers.)

A way of summarizing the main point of this section is that for real numbers,

A. division by 0 is not meaningful,
or
B. the square root of a quotient is the quotient of the square roots?

Chapter 28: The Cartesian Coordinate System



554. The Cartesian coordinate system is very important in mathematics, for many reasons. But one important way that it's used is to tell people where things are, and how to get from one place to another.

Suppose we want a system to locate any point on a flat surface, or a plane. If we draw on this plane two perpendicular lines called axes, we can then locate any point on that plane by giving the distance of the point to each of the two axes. (The word axes is the plural of the word axis.)

Do you see the heavy dark lines in the picture above, the ones that cross each other a little below and to the left of the center of the picture? We call the horizontal one, the one that goes from left to right, the x-axis. Another word for x-axis is the abscissa. We call the vertical one, the one that goes from down to up, the y-axis. Another word for the y-axis is the ordinate.

The x-axis is

A. horizontal,

or
B. vertical?

555. We call the place where the x-axis and the y-axis cross, the origin. This is our imaginary starting point, for describing where any other point is. The other lines in our picture, that run parallel to the x- and y- axes, help us to measure how far points are from the x- and y- axes. We can tell where any other point on the plane is by telling how many units left or right, and how many units up or down, you go from the origin to get to that point.

Do you see the smiling face in the picture? How would you tell someone how to get to that smiling face, from the origin? You could say, "Start at the origin. Go two units (or two blocks) to the right, and then three units up." This makes this point two units to the right of the y-axis, and three units above the x-axis. There is one, and only one, point that is that pair of distances from the two axes.

The main idea of this section is that

A. You can locate any point on the flat surface by saying how far to the right and how far up you would go from the origin,
or
B. the Cartesian coordinate system was invented by Rene Descartes?

556. The point that locates the center of the nose of the smiling person in the picture is called 2,3. This means start at the origin, go two units to the right, and then go three units up. We name points by giving two numbers, separated by a comma. The number that tells how many units to the right is called the x coordinate, and the one telling how many units up is called the y coordinate. You can remember that the x coordinate comes first when we name the points, by recalling that x comes before y in the alphabet.

A point that is 5 units to the right of the origin and 1 unit above it would be called

A. 5,1
or
B. "5 right 1 up" ?

557. It’s important to realize that with our coordinate system, we’re locating points and not squares. A point is represented by a little dot, but mathematicians imagine a point as having no height or width at all. It is the place where two lines cross each other.

It’s convenient, when telling where things are, to talk about points rather than squares. How far do you go from the origin to get to a certain square on our picture? Well, that depends on whether you want to go to the middle of the square, or the upper left corner, or the lower right, and so forth. When we talk about points, these sorts of questions are not a problem. Because a point has no height or width, there is only one place on a point that you can go to!

The main idea of this section is that

A. the coordinate system we’ve talked about so far applies only to flat surfaces, not 3-dimensional space,
or
B. the coordinate system tells the location of points rather than squares, and this is a good way to set up the system?

558. You’ll recall that we defined the x-coordinate as how many units to the right of the origin a point is, and we defined the y-coordinate as how many units above the origin the point is. What happens if a point is to the left of the origin, or below it?

This is one of many times when negative numbers come in handy. A negative number for the x-coordinate means that you go to the left of the origin, and a negative number for the y-coordinate means that you go down from the origin. Do you see the point with coordinates –3, -1 on our graph? To get to this point, you go three units to the left of the origin, and one unit down. The negative numbers mean left and down, just as the positive numbers mean right and up.

Do you see the point –4, 5? To get here, you go 4 units to the left and 5 units up. How about the point 5, -2? To get there, you go 5 units to the right and 2 units down.

The main idea of this section is that

A. to described distances to the left of and below the origin, we use negative numbers for the x and y coordinates, respectively,

or
B. the vertical distance between two points is called the rise, and the horizontal distance is called the run?

559. The various combinations of negative and positive coordinates divide our graph into four sections called quadrants. These are the same four sections that are created by the crossing of the x-axis and the y-axis. The upper right section has positive coordinates for both x and y; it’s called quadrant 1. The upper left section has a negative x coordinate, and a positive y-coordinate; it’s called quadrant 2. The lower left section has negative coordinates for both x and y; it’s called quadrant 3. And the lower right section has a positive x-coordinate and a negative y-coordinate; it’s called quadrant 4. So to number the coordinates, you start at the upper right and go counter-clockwise, numbering the 4 quadrants 1, 2, 3, and 4.

The main idea of this section is that

A. One of the main uses of the coordinate system is to make pictures of functions, or
B. We name the four sections of the plane quadrant 1, 2, 3, and 4, starting in the upper right and going counterclockwise?

560. When we think about the relation of two points to each other, it’s useful to define two more words: the run, and the rise. The run is the horizontal part of the trip between the two points, and the rise is the vertical part. An example will show you what I mean by this.

Please look at the picture at the beginning of this chapter. Let’s imagine that the surface were big, and we are walking from –3, -1, to 2,3. Suppose that the lines are walkways, and we are not allowed to cut across the squares. There are lots of ways we could go from the first point to the second. We could first go horizontally, and head from –3,-1 to 2,-1. This trip is five units long. Then we could go the rest of the way vertically, from 2, -1 to 2, 3. This part of the trip is four units long. Thus the run is 5 and the rise is 4.

On a trip between two points, the rise is the

A. horizontal part,
or

B. vertical part?

561. On our trip between –3, -1 and 2, 3, we could go a different route. We could first walk 5 units up, to –3,3. Then we could walk 5 units to the right, which gets us to our destination, 2,3. Again the run is 5 and the rise is 4. And we could take all sorts of zig-zaggy routes, where we would for example start with one unit up, then go one to the right, and so forth. But as long as we always move either up or to the right, and don't backtrack, and plan our trip so that we end up on 2,3, we will have always travelled 5 units to the right and 4 units up.

The point of this section is that

A. the ratio of the rise to the run is called the slope of a line joining two points,
or
B. the rise and run come out the same no matter what order you take your horizontal and vertical jumps in, as long as you don't backtrack?

562. Suppose our starting point happens to be a special point, namely the origin. To get from the origin to 2, 3 you would walk two units right and 3 units up. So the run is just the x-coordinate, and the rise is the y-coordinate. To get from the origin to –3, -1, we would go 3 units to the left (which we can think of as –3 units to the right) and 1 unit down (which we can think of as –1 unit up)! Thus again, the x-coordinate is the run and the y-coordinate is the rise. The x and y coordinates of any point are the run and the rise, respectively, if the starting point is the origin.

This section brought us to the conclusion that

A. the straight-line distance between any two points can be computed from the rise and the run,
or
B. the coordinates of any point are just the run and rise, respectively, on a trip from the origin to that point?

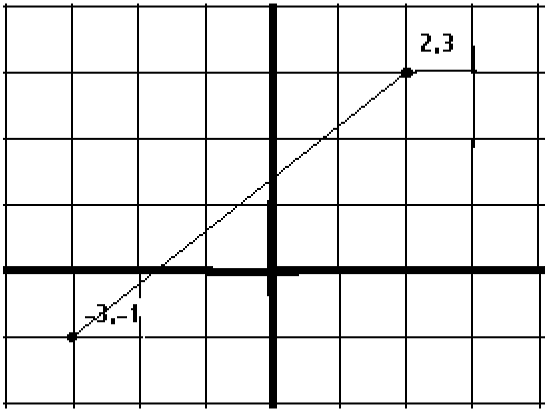
563. What do you think the coordinates of the origin are? If we were giving directions on how to get to the origin from the origin, we would say, "First walk 0

units to the right, and then walk 0 units up! You can't miss it!" Because those two distances are 0 and 0, the coordinates of the origin are 0,0.

The purpose of this section was to

A. explain how to compute the straight-line distance between two points given the rise and the run,
or
B. explain why the coordinates of the origin are 0,0?

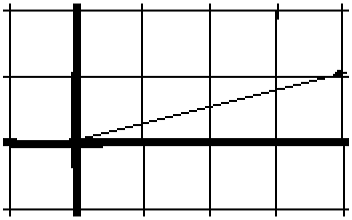
Slopes of lines



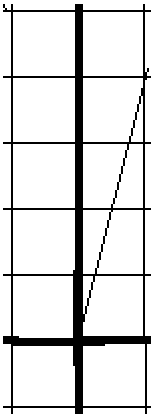
564. In the picture above, I've drawn a line between –3,-1 and 2,3. We can describe the steepness of lines by defining something called *slope*. The slope of a line is the ratio of the rise to the run, or the rise divided by the run. For a trip from –3,-1 to 2,3, the rise is 4 and the run is 5, because we go 5 units right and 4 units up. So the slope is 4/5. What if we took the trip in the other direction, from 2,3 to –3, -1? We would run –5, by going 5 units to the left (-5 units to the right). We would rise –4, by going 4 units down (-4 units up). So the slope would be –4/-5, which (when we multiply top and bottom of that fraction by –1) comes out to 4/5. Just as in this example, with all other pairs of points, the slope you get doesn't depend on which point on the line you start at.

The main purpose of this section was to

A. define the slope of a line,
or
B. tell that a negative number divided by a negative number gives a positive number?



565. Above is a picture of a line segment that rises 1 unit while it runs 4 units. Its slope is 1/4. The line isn't as steep as the one below. It rises 4 while running 1; its slope is 4.



Thus for lines that slope from lower left to upper right, the steeper is the slope, the larger is the value of the slope. The values get higher and higher as you get closer and closer to a vertical line. For a horizontal line, there is 0 rise for any run that you pick, so the slope is 0.

Which type of line has a slope equal to 0?

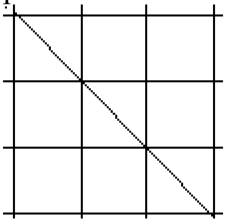
A. a horizontal line,
or
B. a vertical line?

566. What sort of slope does a vertical line have? Such a line rises but does not run. For a rise, say, of 1 unit, there is a run of 0 units. So the slope would be the rise divided by the run, or 1/0. But since division by 0 "is not allowed," or has an undefined answer, the slope of a vertical line is undefined.

What's the major point of this section?

A. The slope of a horizontal line is 0,
or
B. the slope of a vertical line is undefined?

567. What about the slope of a line that goes, not from lower left to upper right, but from upper left to lower right? For example, what is the slope of the line pictured below?



For each one unit that the line goes to the right, it moves one unit down, or it rises –1 unit. Or we could say that for every 1 unit that it goes up, it goes to the left one unit, or goes to the right –1 unit. Either way, the rise divided by the run comes out to –1. It turns out that ALL lines that slope downward as you go from left to right have negative slopes, and all lines that slope upward as you go from left to right have positive slopes.

The main purpose of this section was to

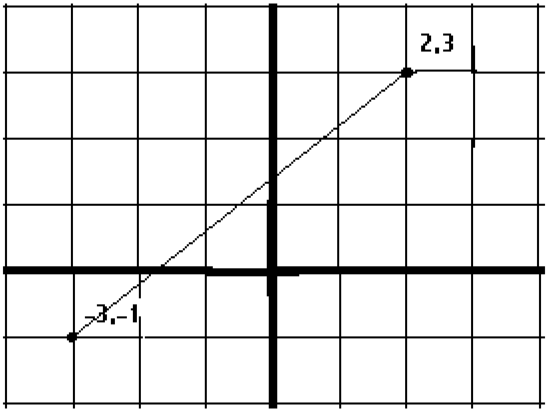
A. tell about negative slopes,
or
B. tell about a line that fell the same number of units as it ran?

568. So as a line with a negative slope gets steeper and steeper, that is closer and closer to vertical, the slope gets to be a negative number with a larger and larger absolute value. For example, a line with a slope of –1000 would be nearly vertical, just as a line with a slope of +1000 would be nearly vertical! The one with the negative slope would be slanting down as you move from right to left, and the one with the positive slope would slant up as you move from right to left.

One of the surprising facts that this section tells is that

A. a line with slope of –1000 would probably look almost like a line with a slope of 1000,
or
B. lines with negative slopes point downward as you go from left to right?

569. So far we've found runs and rises by counting the number of units on a graph that we travel out or up as we move from one point to another. But there's a much faster way. Let's look at the trip from –3,-1, to 2,3. How far, horizontally, is it from –3 to 2? We could count the jumps, but it's faster just to subtract the first value from the second. 2 minus –3 is the same as 2+3, or 5, and this is the same number we get by counting. Likewise, how far up is it from –1 to 3? We can subtract 3 –(-1) and get 4. So in other words, we get the run by subtracting the x values and the rise by subtracting the y values.



The main point of this section is that

A. since the line from –3,1 to 2,3 goes from lower left to upper right, the slope is positive,
or
B. you can get the rise and run by subtracting, as well as by looking at a graph and counting?

570. If we think of taking a trip from –3,-1 to 2, 3, we run 5 and rise 4. On the other hand, if we think of taking a trip from 2,3 to –3,-1, we run –5 and rise –4. Thus the rise and run are of opposite signs, depending on what sign you think of as the starting point.

But when you compute the slope by dividing the rise by the run, you get the same answer no matter which point you consider the starting point. In our example, 4/5 is the same as –4/-5. We can consider either point to be the first one, and we get the same slope, as long as we consider the same point as the starting point for both the x values and the y values.

The main idea of this section is that

A. the slope of a line going from upper left to lower right will always be computed by dividing two numbers, one of which is positive and the other of which is negative,
or

B. when you are figuring out the slope between two points by subtracting, you can count either point as the first and the other as the second, and you'll get the same slope, as long as you do the same thing for both the rise and the run?

571. Here's a way of saying the same things we've been saying about computing slopes by subtracting, but saying them in mathematical symbols. Suppose we have two points. We call the first one x_1,y_1 (which you read, "x sub 1, y sub 1) and the second one x_2, y_2. The formula for slope is:

slope = $(y_2-y_1)/(x_2-x_1)$.

This is just the difference in y values over the difference in x values.

The purpose of this section was

A. to give a formula for the slope, in mathematical symbols,
or
B. to introduce the concept of slope?

572. Let's interrupt the discussion of slope for a minute to tell you something that will be useful to remember in your journey through mathematics. You remember that the opposite of a number is the same absolute value, only with the opposite sign. So the opposite of 6 is –6, and the opposite of –3 is 3, and the opposite of x is –x.

Here's the useful fact to remember. When you have a difference, that is a first number minus a second, the opposite of that difference is the second number minus the first.

It's much easier to remember when you put it in mathematical symbols. The opposite of a-b is b-a. The opposite of x-y is y-x. The opposite of 5-3 (or 2) is 3-5 (or –2).

Let's prove this. Suppose we start with a-b. To find its opposite, we multiply it by –1, to get –1(a-b). Now let's use the distributive law to multiply both terms by –1: we get –1*a + -1*-b. This is the same as –a +b. Now using the commutative law, we can change that to b-a. Thus we've proved that the opposite of a-b is b-a.

Which is an example of the rule we talked about in this section?

A. 10-7 is the opposite of 7-10,
or
B. since 7*5=35, 35/5=7?

573. Suppose we take the slope, which we defined as $(y_2-y_1)/(x_2-x_1)$, and we multiply numerator and denominator by –1. Using the fact we just discussed, we would get a formula that looks like this:

slope=$(y_1-y_2)/(x_1-x_2)$.

This formula is the same as the first one, only with the order of subtracting reversed for both the rise and the run. This helps us feel confident that we can define either point as the first one, and we'll get the same slope.

This section

A. spoke about the use of Cartesian coordinate systems to make pictures of functions,
or
B. tended to prove that you get the same slope no matter which point you call the first one?

574. Now, since you have done so much reading about slopes, you are entitled to get at least a beginning of an answer to the question, why are slopes so important? The answer is that slopes tell us how much one variable changes when we change another, and this type of question is central to all sorts of scientific issues. For example: if you change the amount of studying that students do by thirty minutes a day, how much do you change how much they learn? If you change the amount of pesticides used in a certain region, how much do you change the rate with which the residents of the region get Parkinson's disease? If you increase the average cooperation score among students in a school district, how much do you change the rate of depression in the students? If you increase the amount of alcohol a driver has inside him, how much do you change his chance of having a wreck? If we increase government spending on a certain project by a certain number of dollars, how much do we increase the number of people with jobs? All of these questions, and millions more, can be answered in terms of slopes.

The point of this section was that

A. mathematicians all over the world have agreed to call the slope the rise over the run rather than the run over the rise,
or
B. slopes are so important because lots of things we want to know about the world have to do with how much one variable will change if we change another?

Chapter 29: Using Cartesian Coordinate Systems to Make Pictures of Functions

575. We've spoken earlier about functions. We imagined a function as a machine, where if you put a certain number in, you get a certain number out, according to a set rule. Function rules can be communicated in four ways:
1. We can explain in words and sentences what the relation is between what we put in, and what we get out.
2. We can present a table, with typical numbers that we put in, and the corresponding numbers we get out.
3. We can present an equation, or a formula, that tells the relation in mathematical symbols.
4. We can present a graph. Usually the number we put in is given on the x-axis, and the number we get out is on the y-axis.

A summary of this section is that

A. A function rule tells how an output number depends on an input number, and those rules can be communicated by words, tables, formulas, or graphs,
or
B. functions are so important because of people's need to predict one thing, knowing another?

576. Suppose someone gets paid \$12 per hour for working. If they work one hour, they get \$12. If they work two hours, they get \$24. If they work 3.5 hours, they get 3.5*12 dollars, or \$42. If they work 100 hours, they get \$1200 dollars. The amount they get paid is a function of how many hours they work. If we want to put the function rule into words, we simply say, "You get paid \$12 for each hour you work." If we want to make a formula, or equation, that expresses that function rule, we would say
$y=12x$
where x is the number of hours you work, and y is the number of dollars you get paid.

If we want to make a table with some of the values of this function, it might look like this:

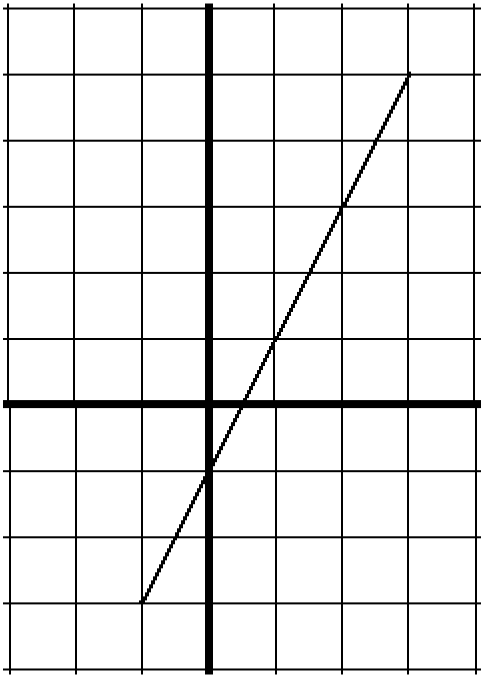
x	0	1	2	3	5	10	100	1000
y	0	12	24	36	60	120	1200	120000

The section above illustrated which of the following 3 ways of communicating a function?

A. words, graphs, and tables,
or
B. words, tables, and formulas?

577. Let's look at the picture of a different function. The origin, or the point (0,0), is where the heavy lines cross. Every point on the diagonal line is a pair of values for x and y. What are some of those values? The places where the line crosses the intersections of the lines on the graph are the points where there are integer values for both x and y. One of those points is 1,1 (to get to this point, start at the origin, and go one to the right and one up.) Another of these points is 2,3.

If we pick 1 as the x value to put into the function machine, we get 1 as the y value coming out. If we pick 2 as the x value to put in, we get 3 as the y value coming out. The diagonal line on our graph hooks up one value of y for each of many more values of x. Thus the line gives a picture of a function.



The main point of this section is that

A. The line on our graph hooks up one value of y for many different values of x, and thus is the picture of a function,
or
B. lines such as the one pictured above, that slant from lower left to upper right, have positive slopes?

578. Let's make a table of some of the values of the function pictured on the previous page. Here are some:

x	-1	0	1	2	3
y	-3	-1	1	3	5

How can you figure out an equation, or formula, for this line? There are some foolproof ways of figuring out the equation, given a table of values like this. You can even figure out the formula for a line when you know only two points! But rather than figuring it out now, I'll just tell it to you. It's y=2x-1 .

Let’s check out that formula for some of the points in the table. Does 1,1 obey that formula? We check it out by saying y= 2x-1, then substituting 1 for x to get y=2*1 – 1. From this we get y=1. So when x=1, y=1, and the point 1,1 does obey the formula! Let’s check it out with 3,5. y=2x-1; substituting 3 for x we get y=2*3-1, or y=5. Thus the point 3,5 obeys the formula too!

If we wanted to put the function into words, we would say something like, “To get y, we first double x and then subtract 1 from what we get.”

From the information in this section you can infer that

A. if you want to check whether a certain point goes with a certain formula, you substitute the x value for x in the formula, and see if the y value you get is the same as the y value for the point,
or
B. if you want to find the equation that goes with a certain table, you start with figuring out the slope, by seeing how much y increases for a certain increase in x?

579. The function y=2x-1 could have very large or small values, for example (1000, 1999), or (–2000, -4001). However, the graph of our function on the previous page doesn’t go on forever in both directions. As you go to the left, it stops when x=-1. You don’t see a point on this picture for when x=-1.1, or –2, or any smaller values. Similarly, as you go to the right, the graph stops when x=3. You don’t see points for x=3.1, 3.9, 6, and so forth. All the values of x between, and including, -1 and 3 are permitted for this graph. We call the set of numbers such that $-1 \le x \le 3$ the *domain* of the function. The domain is the set of all input values, or x-values, that are permitted.

Because we restrict the x values we put in, we also restrict the y values we can get out. In our picture, y can take on any value between and including –3 and 5, but no others. The set of numbers such that $-3 \le y \le 5$ is the range of the function. The range is the set of output values, or y-values, that are permitted.

So if we think of a function as a machine where you put certain numbers in and get certain numbers out, the domain is the set of numbers we’re allowed to put in, and the range is the set of numbers we can possibly get out.

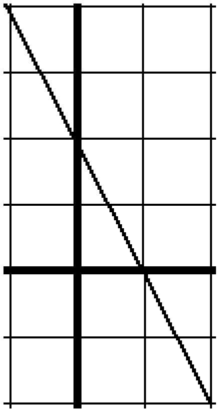
The purpose of this section was to

A. explain why the place where the graph crosses the y-axis is a special point,
or
B. explain the meaning of the ideas of domains and ranges of functions?

580. When you have a function whose graph is a straight line, we call the function a linear function. We've already talked about the slopes of linear functions. There's another special part of the graph of a linear function: it's called the y-intercept. The y-intercept is the y-value of the point where the line crosses the y-axis. Or in other words, the y-intercept is the other coordinate of the point where x is 0.

Please look back at the line that is the graph of y=2x-1. What's the y-intercept? You see the heavy vertical line; that's the y-axis. The graph of the function crosses it right at 0,-1. So –1 is the y-intercept.

Below is the graph of another linear function. It crosses the y-axis at the point 0,2. So for it, 2 is the y-intercept.



The purpose of this section was to

A. define the x-intercept as the point on the graph where y is 0,
or
B. define the y-intercept as the point on the graph where x is 0?

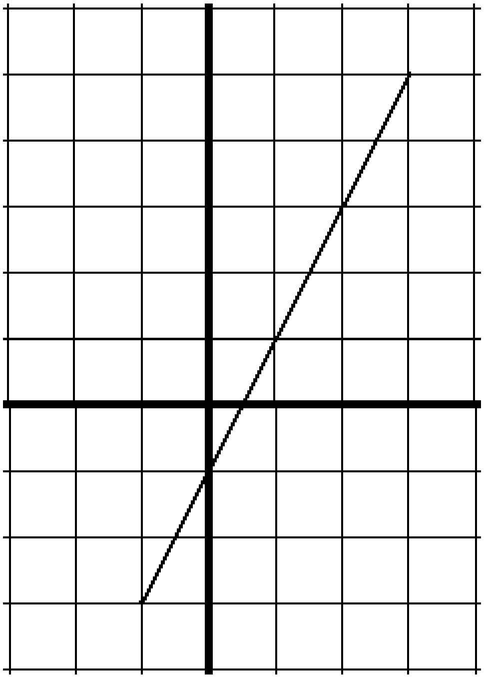
581. Below is another copy of our y=2x-1 graph. What's the slope of this line? We can find this slope from the graph, by seeing what the ratio is of the rise to the run. Let's start at the point 1,1. If we go one unit to the right (a run of one unit) and two units up (a rise of two units) we are at another point on our line. So dividing 2 by 1 gives a slope of 2.

We could also find the slope using our formula, slope = $(y_2-y_1)/(x_2-x_1)$. Using the points 3,2 and 1,1 would give a slope of (3-1)/(2-1) or 2/1, or 2.

So the formula for the line is y=2x-1, the slope is 2, and the y intercept is –1. Do you notice something about how the slope and the intercept correspond to the equation? The slope is the number that is multiplied by x, and the y intercept is the number that you then add (in this case, you add –1).

It turns out this is always true. Whenever you have an equation of the form y=mx + b,

when you graph that function, m is the slope and b is the y-intercept! For this reason, y=mx+b is called "slope-intercept form" for a linear function.



The main idea of this section is that

A. when you have any equation in the form y=mx + b, the slope is m and the y-intercept is b,
or
B. when you have the equation y=2x –1, the slope is 2 and the y-intercept is –1?

582. Let's talk for a minute about the correspondence between a picture of a function (in a graph) and the equation for a function. The slanting line above is a picture of the function, y=2x-1. This means that for every point on that line, if you get the x and y coordinates, those x,y pairs will make the equation y=2x-1 true. For example: -1, -3 is one of the points on our line. Let's test it by substituting –1 for x and –3 for y into our equation, and see if it makes the equation true.

y=2x-1
-3=2*-1 –1
-3=-2-1
-3=-3

So the point –1, -3, does satisfy our equation, and it should indeed be on the line that is the graph of that equation!
We can do the same thing with any or all points on our graph. It works for values where one or both values for x and y are not integers. Do you see that the point .5, 0, appears to be on our line? Let's check this out, by substituting the two values into y=2x –1.

y=2x-1
0=2*.5-1
0=1-1
0=0

It works, so .5, 0 is on our graph.

The major point this section is trying to make is that

A. equations involving x and y raised to the first power only always come out to be straight lines,
or

B. if a point is on a graph of a function, that means that substituting the x and y coordinates of that point into the equation for the function makes the equation true?

583. If someone gives you an equation like $y=5x +3$ and asks you, “What are the slope and the y-intercept,” the answers are easy. The slope is 5 and the y-intercept is 3. But sometimes you have to do a little algebra and add, subtract, multiply, or divide both sides of the equation by certain numbers to get the equation into the $y=mx + b$ form. Here’s an example. Suppose someone asks for the slope and y-intercept that results from the following equation:

$4x + 2y - 16 = 0$.

First we can add 16 to both sides of the equation. We get

$4x + 2y = 16$.

Next we subtract $4x$ from both sides of the equation. We get

$2y = -4x + 16$.

Finally, we divide both sides of the equation by 2. We get

$y= -2x + 8$.

Now, we have an equation in the $y=mx + b$ form. We look at the number multiplied by x and say that the slope is -2; we look at the number added and say that the y-intercept is 8.

The point of this section was to tell you

A. how to solve two equations in two unknowns,
or
B. how to find the slope and intercept of an equation in y and x, by getting the equation into the form $y=mx+b$?

584. Here's another problem to solve. Suppose you have an equation, where you know the slope, but the y-intercept is what you have to figure out. And suppose you also know one point that satisfies that equation, or one point on the line that is the graph of that equation. How can you figure out the y-intercept? Here's an example of the type of problem I'm talking about:
The slope of a line is 4. What's the equation of that line, if it goes through the point 2,6?
Here's how we solve this. If the slope is 4, the equation of the line must take the form

$y=4x + b$.

If we knew what b was, we would know the equation of the line. How can we find b? We can do it by taking the equation above, substituting 2 for x and 6 for y, and solving for b. In other words, we know the correct b is the one that makes $6=4*2 + b$, and we just have to figure out what that b is. We use algebra to solve this equation:
$6=4*2+ b$
$6=8+ b$
$-2=b$

So we've just found out that $b=-2$. Therefore, the equation of the line is

$y=4x – 2$.

This section told you how to solve a certain type of problem in which you

A. find the equation of a line, given the slope of the line and one point,
or
B. find the slope of a line, given the equation for it?

585. It turns out that any time you have an equation of the form $y=mx+b$, or of the form $ax + by +c = 0$, the graph of the equation is a straight line. In other words, any equation involving x and y raised only to the first power (that is, not squared or cubed and so forth) you get a straight line. Why is this? First, you can change an equation of the form $ax +by +c = 0$ into one of the form $y=mx +b$ by just using the golden rule of equations, as we illustrated a couple of sections ago.

Second, the slope of the graph at any point is always m. But if you think about it, a straight line is one where the slope doesn't change, whereas a curved line is one where the slope does change.

The main point of this section is that

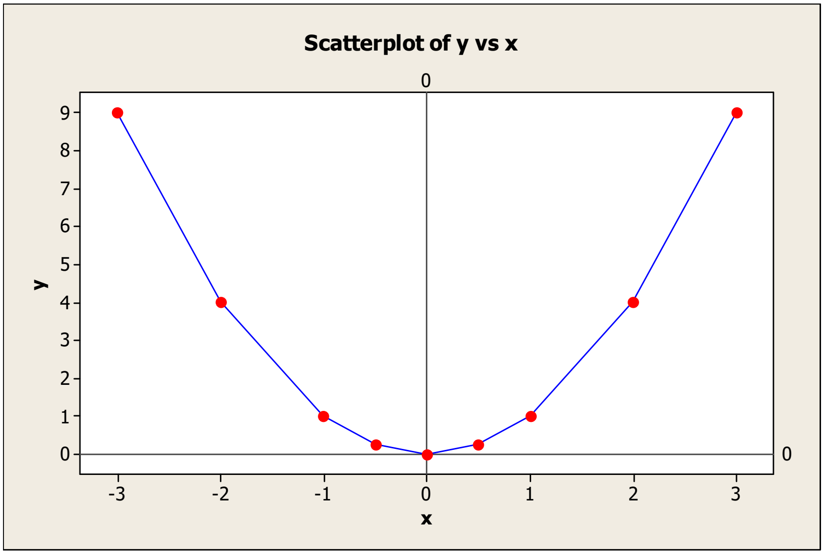
A. functions involving x^2, when graphed, give curved lines,
or
B. all equations of the form $y=mx+b$, or $ax + by + c=0$, when graphed, give straight lines?

586. Let's look at the picture of a function that yields a curved line. Let's think about the equation $y=x^2$. If we make a table of some of the values, it might look like this:

x	0	± 1/2	± 1	± 2	± 3
y	0	1/4	1	4	9

In this table, the symbol ± means "plus or minus." Thus when x is +2, y is 4; when x is –2, y is also 4.

If we graph these values, and connect the points by line segments, we get the following:



Even with this few points, the graph looks curvy. If we were to graph all the points that satisfied $y=x^2$, we'd get a smooth curve that is sort of u-shaped, very much like the shape above, that is called a parabola.

One of the facts this section went over is that

A. the graph of $y=x^2$ produces a curved line called a parabola,
or
B. the graphs of $y= x^3$ and $y= x^4$ also produce curved lines?

587. What about equations like $y=3x^2 +2x –3$, or $y=8x^2 -17x +14$? It turns out that the pictures of these functions are parabolas also. They are located at different places on the coordinate system than $y= x^2$ is, and they may be skinnier than our $y=x^2$ parabola, and they may be turned upside down, but they still are unmistakably parabolic in shape. In fact, any equation of the form $y= ax^2 +bx + c$, where a and b and c are constants (that is, ordinary numbers like 6 or 8.2 or –pi), when graphed, makes a picture that is a parabola.

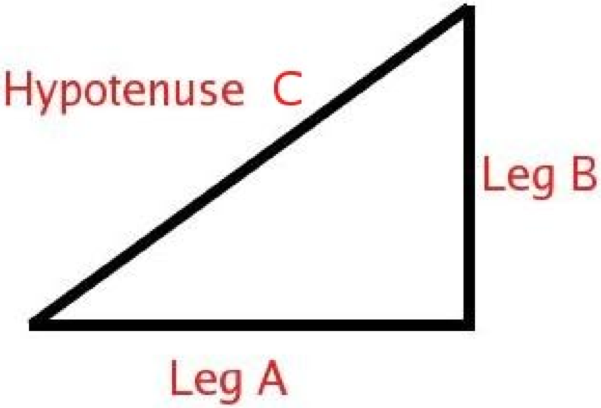
The point of this section was to tell you that

A. all equations where y is a function of something times x^2 and something times x and a constant, when graphed, come out to be parabolas,
or
B. parabolas where the coefficient of x^2 is negative are upside-down versions of the parabola we pictured before?

Chapter 30: The Pythagorean Theorem and the Distance Formula

588. The Pythagorean Theorem is a very important mathematical relationship. Before we talk about it, let's get familiar with the words that it uses.

A right triangle is any triangle with a right angle. In the picture below, the right angle is in the lower right of the drawing. The side of the triangle that is opposite the right angle is called the hypotenuse. It's always the longest side of a right triangle. The other two sides of the triangle, the two that are at right angles with each other, are called the two legs of the triangle. In the drawing below, as in many other that have to do with the Pythagorean Theorem, the two legs are called A and B, and the hypotenuse is called C, although if we wanted to we could label them with any other letters we want.



This section

A. presented the Pythagorean Theorem,
or
B. aimed to get us familiar with the words involved in the Pythagorean Theorem?

589. Here's the Pythagorean Theorem in words: The square of the hypotenuse is equal to the sums of the squares of the legs. Or in symbols, if we call the two legs a and b and the hypotenuse c,

$c^2 = a^2 + b^2$.

Why is this important? It allows us to find the third side of any right triangle, if we know the other two sides, and if we know which is the hypotenuse and which are legs. There are many times when you will want to know a third side given the other two sides.

What's a summary of this section?

A. The Pythagorean Theorem, which says that the square of the hypotenuse is equal to the sums of the squares of the legs, allows you to solve problems where you know two sides of a right triangle and you want to find the third.
or
B. The Distance Formula is a direct consequence of the Pythagorean Theorem.

590. Let's look at an example of a problem you would solve with the Pythagorean Theorem. The two legs of a right triangle are 3 and 4 centimeters long. How long is the hypotenuse?

If c is the length of the hypotenuse, then

$c^2 = 3^2 + 4^2$

$c^2 = 9 + 16$

$c^2 = 25$

Now, we take the square root of both sides of this equation. We get that c must be 5 cm.

This section

A. gave a proof of the Pythagorean Theorem,
or

B. gave an example of a problem solved by the Pythagorean Theorem?

591. Here's another problem that is only slightly different. In the last section the two legs were known, and we solved to find the hypotenuse. In the following problem, the hypotenuse and a leg will be known, and we'll solve to find the other leg. Here's the problem: The hypotenuse is 13 units, and a leg is 5 units. What's the length of the other leg? Substituting 13 for c and 5 for a in our formula, we get

$13^2 = 5^2 + b^2$

$169 = 25 + b^2$

When we subtract 25 from both sides, we get

$144 = b^2$

And when we take the square root of both sides we get

$12 = b$

So our answer is that the missing leg was 12 units long.

This section

A. explained why the hypotenuse is always the longest side of a right triangle,
or
B. gave an example of a problem solved by the Pythagorean Theorem when a leg and a hypotenuse are given and the other leg is to be found?

592. In the last two examples, the answers have come out as integers. In triangles with sides of 3, 4, and 5, and 5, 12, and 13, the Pythagorean Theorem is satisfied with whole numbers. Sets of three numbers that work like this are called Pythagorean triples. Most right triangles don't have sides that come out as integers. If the legs are 1 and 2, the hypotenuse is the square root of 5. If a leg and the hypotenuse are 3 and 6, the other leg is the square root of 27, or 3 times the square root of 3.

When you take tests having to do with the Pythagorean Theorem, often the test makers don't want to mess with irrational numbers. It's easier to make answers that are integers. So often the problems are made to include 3, 4, 5 triangles or 5, 12, 13 triangles. Another favorite is 6, 8, 10 triangles, another Pythagorean triple.

But be careful – there are many exceptions to this.

The main point of this section is that

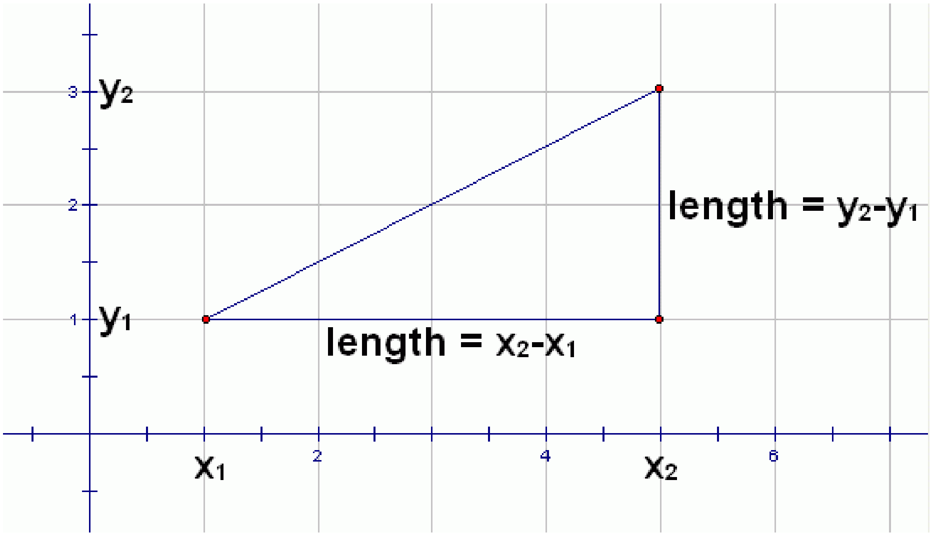
A. Test questions dealing with the Pythagorean Theorem often have the triangles have sides of lengths that come out as whole numbers, such as 3, 4 and 5; 5, 12, and 13; and 6, 8, and 10.

or

B. Some evidence suggests that the Babylonians knew about the Pythagorean Theorem at least 1000 years before Pythagorus.

593. Now, with the Pythagorean Theorem in mind, let's go back to our Cartesian coordinate system. The Pythagorean Theorem leads to something called the distance formula. Suppose that you have two points, let's say 1, 1 and 5, 3. How far apart are those two points? As the drawing below shows, we can think of the straight line distance between 1,1, and 5,3 as the hypotenuse of a triangle. Counting the squares reveals that the legs are 4 units and 2 units long. We could also have gotten the horizontal distance by subtracting 1 from 5, and the vertical distance by subtracting 1 from 3. Then we use the Pythagorean Theorem with two legs of 4 and 2, and we get that the hypotenuse is the square root of 20 units long, or about 4.5 units long.

This is the sort of problem the distance formula is meant to solve. If we want to make the problem general, we can say that the two points are x_1, y_1, and x_2, y_2. The two legs are the horizontal distance between the two points, or $x_2 - x_1$, and the vertical distance between the two points, or y_2-y_1. We use the Pythagorean Theorem with the lengths of those two legs to find the distance between the two points.



This section tells us that

A. The distance formula just uses the Pythagorean Theorem to find the distance between two points, as long as we keep straight that the two legs of the right triangle are $x_2 - x_1$ and y_2-y_1.
or
B. The numbers that satisfy the equations we get when using the Pythagorean Theorem can be positive or negative, but only positive numbers make sense when we're talking about distances?

594. So the distance between two points x_1, y_1, and x_2, y_2 is the following:

d=

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

In words, the distance is the square root of x sub 2 minus x sub1 squared plus y sub 2 minus y sub 1 squared . And this is the distance formula, which comes directly from the Pythagorean Theorem. The difference between the x's is the length of the horizontal leg, and the difference between the y's is the length of the vertical leg.

The purpose of this section is to

A. present the distance formula,
or
B. to give an example of the type of problem where the distance formula is used?

595. Let's do a distance formula problem before we finish this chapter. Here it is: what's the distance between 2,3 and 7,15?

To get the length of the horizontal leg of the right triangle, we subtract the x values: 7-2 gjves us 5. To get the length of the vertical leg, we subtract the y values: 15-3 gives us 12. Now we square 5 and 12, getting 25 and 144, add the results together, getting 169, and take the square root, getting 13. So the answer to our problem is 13 units. You will notice that like the test makers I mentioned earlier, I didn't want to mess with harder computations, so I used a Pythagorean triple in this problem!

The purpose of this section was to

A. give an example of how the distance formula is used in a problem,
or
B. explain that the midpoint of the line joining two points has coordinates that are the average of the x values and the average of the y values, respectively?

Chapter 31: Multiplying Binomials And The Foil Rule

596. An expression like 2x or $3x^2$ or 10y is called a monomial. An expression like a+b or 2x+3y or 4t+7 is called a binomial. For binomials, there are two numbers (called "terms") added together. A trinomial has three terms added. Examples of trinomials are expressions like $x^2 + 2x + 1$, or $3x^2 + 2xy + 4 y^2$.

The purpose of this section was

A. to tell how to multiply binomials by each other,
or
B. to give examples of monomials, binomials, and trinomials?

597. How do you multiply two binomials by each other? For example, how do you multiply (x+2) by (x+3)? Doing this involves the distributive law. You remember that the distributive law says that a(b+c) is ab + ac. In words, when you multiply a number by a sum, you can multiply that number by both of the addends and then add the products together. You often use the distributive law when you are multiplying something by two numbers added together in parentheses. So for example, 2*(3+4) is equal to 2*3 + 2*4. It checks out, because 14 equals 6 + 8.

Remember that the order in which we multiply doesn't make a difference. So we could also state the distributive law by saying (b+c)a = ba+ca.

This section

A. raised the question of how to multiply binomials, and then reviewed the distributive law,
or
B. gave a complete explanation of how to multiply binomials?

598. Now let's multiply (x+2) by (x+3) by using the distributive law three times. When we use it the first time, we get rid of the parentheses around (x+3):

(x+2)*(x+3) = (x+2)*x + (x+2)*3.

Now, let's use it twice more, to get rid of the parentheses around (x+2).

$$=x*x +2*x + 3*x + 2*3$$

$$= x^2 + 2x +3x +6$$

$$= x^2 + 5x +6$$

This section

A. showed that it doesn't make a difference whether the sum in the parenthesis comes before or after the number you're multiplying it by, when you're using the distributive law,
or
B. gave an example of how to multiply two binomials?

599. Let's do the same thing again, with another pair of binomials: (a+b)(c+d). When we use the distributive law the first time, let's get rid of the parenthesis around (c+d).

$(a+b)(c+d)=(a+b)c + (a+b)d$

Then we use the distributive law to get rid of the remaining parentheses:

$$= ac +bc+ad+ab$$

Let's use the commutative law to rearrange these a little bit:

$$=ac +ad +bc + bd.$$

So this is another way of expressing the answer to the product of two binomials. Why is this called the FOIL rule? Look again at (a+b)(c+d). We can name a and c the "first" terms of the two binomials; a and d the "outside" terms because they are farthest from the center of the whole expression; b and c the "inside" terms because they are closest to the center, and b and d the "last" terms, because they come last in their binomials.

Now let's look again at the answer we got: $ac+ad+bc+bd$. This is the product of the First terms, plus the product of the Outsides, plus the product of the Insides, plus the product of the Lasts. The first letters of First, Outsides, Insides, and Lasts spell the word FOIL, so our procedure for multiplying binomials is called the FOIL rule.

What's a summary of this section?

A. The commutative law of addition says that when we add, order doesn't make a difference; the commutative law of multiplication says that order doesn't matter when we multiply.
or
B. When we multiply two binomials, we end up adding the products of the first, outside, inside, and last terms of the binomials.

600. Now let's just multiply binomials, for one more illustration of the FOIL rule. The multiplication is: $(x-3)(x+4)$

Firsts: $x*x = x^2$

Outsides: $x*4=4x$

Insides: $-3*x=-3x$

Lasts: $-3 * 4= -12$

So when we add these, we get $x^2+4x -3x -12$, or $x^2+x -12$.

Chapter 32: Some Words Used in Logic

601. Part of mathematics involves the study of the relationships of various statements to one another; this study is called logic. If we know that something is true, which other statements follow from that, and which don't?

Suppose I make the following statement: "If this animal is a Collie, then this animal is a dog."

This is a special case of a statement where there is a premise, the part that follows the word *if*, and a conclusion, the part that follows the word *then*. If-then statements have the general form, "If p then q," where p and q are any things we want to say.

Which is an idea that is covered in this section?

A. Deductive and inductive logic are two different types of reasoning.
or
B. In the statement, "If an animal is a Collie, then the animal is a dog," "the animal is a Collie" is the premise, and "the animal is a dog" is the conclusion.

602. What does it mean to "negate" a statement? It means to deny the truth of it, usually by using the word "not." So if we negate the statement "The paper is white," we would say "The paper is not white." To negate the statement "John Doe is alive," we would say "John Doe is not alive."

It's a little trickier when we negate statements involving the words all, no, some, never, always, and so forth. If we negate the statement "All sheep are white," we are overdoing it if we claim that "No sheep are white." To make "All sheep are white" untrue, all we need is one nonwhite sheep. So to negate "All sheep are white," we would say, "Not all sheep are white."

Similarly, if I negate the statement "I never sleep well," it would be overdoing it to claim that "I always sleep well." All we would need to say is "I have slept well at least once." If I negate "No children are wise," the counterclaim would be, "At least one child is wise." If I negate "Some people drive too fast," the counterclaim would be, "No people drive too fast."

Which statement would negate the claim, "No porpoises are fish?"

A. Some porpoises are fish,
or
B. All porpoises are fish?

603. Let’s look at three related statements to our original statement about the dog. If we simply reverse the premise and conclusion, we get the *converse* of the original statement. It would go like this: “If an animal is a dog, then it is a Collie.” From this example alone, we can see that the converse does not follow from the original statement. All Collies are dogs, but not all dogs are Collies.

Which of the following two is the converse of “If it rains, there are clouds?”

A. If it doesn’t rain, there are not clouds?
or
B. If there are clouds, it rains?

604. It’s important to realize that if something is true, the converse is not necessarily true. People often mistakenly act as if a statement implies its converse. “If you have been using a certain drug, you will have a positive result on the urine test,” does not necessarily imply that “If you have a positive result on the urine test, you have been using the certain drug.” “If you don’t exercise enough, you will get tired quickly,” does not imply that “If you get tired quickly, you haven’t been exercising enough.”

The author makes the point in this section that

A. The converse of “If p, then q,” is “If q, then p.”
or
B. People sometimes act as if a statement implies its converse, when it does not.

605. Our second type of related statement is called the contrapositive. To form the contrapositive, you reverse the premise and conclusion and negate each one. So the contrapositive of our original statement about the dog is, “If an animal is not a dog, it is not a Collie.” The contrapositive does follow from the original statement. If the original statement is true, the contrapositive will always be true.

What’s the contrapositive of “If a figure is a square, it is a rectangle?”

A. If a figure is a rectangle, it's a square,
or
B. If a figure is not a rectangle, it's not a square?

606. Our third type of related statement is called the inverse. The inverse of our original statement about the dog is, "If an animal is not a collie, it is not a dog." To form the inverse, you negate the premise and conclusion without changing the order. The inverse, like the converse, does not follow from the original statement. The inverse of "If p, then q" is "If not p, then not q."

What's the inverse of "If this is a quarter, it's a coin?"

A. If it's not a coin, it's not a quarter,
or
B. If this is not a quarter, it's not a coin?

607. To summarize: suppose there is an if-then statement, "If p, then q." Then there are 3 related statements:

The converse: "If q, then p." (This doesn't follow.)
The contrapositive: "If not q, then not p." (This does follow.)
The inverse: "If not p, then not q." (This doesn't follow.)

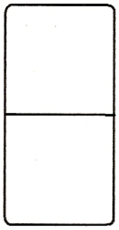
If you can get all this straight in your head, it will probably be good for your everyday reasoning. However, the trouble with these sorts of statements is that so many times in real life, a dogmatic if-then statement is not correct, but what's more correct is a statement of probability. "If you're Republican, you support a higher defense budget," is not true; "Republicans are more likely to support a higher defense budget than are Democrats" is at the time of this writing at least, true. "If you smoke cigarettes, you will get lung cancer," is not true; "If you smoke cigarettes, you will be more likely to get lung cancer," is true. Decision-making in the face of uncertainty, but guesses about probability, is more often what goes on in real life.

A point this section makes is that

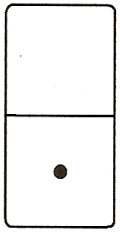
A. Although getting straight what follows from what in if-then statements is probably helpful, real life decisions more often involve statements of the form, “If this is true, that is more likely to be true?”
or
B. The inverse is really the contrapositive of the converse?

Appendix: Pictures of Dominos

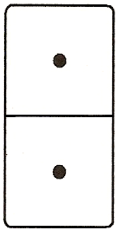
1



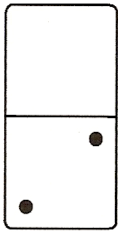
2



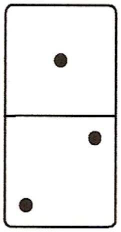
3



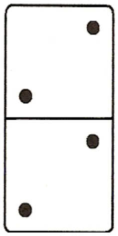
4



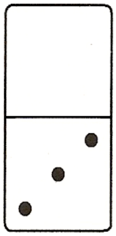
5



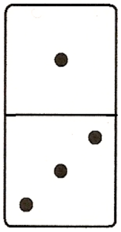
6



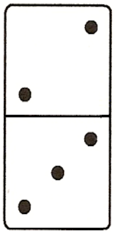
7



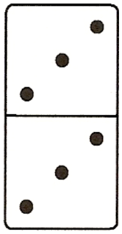
8



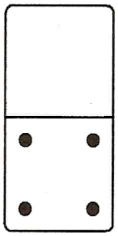
9



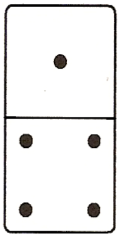
10



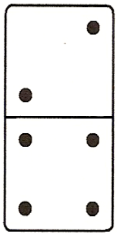
11



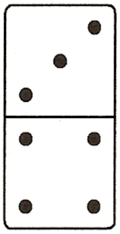
12



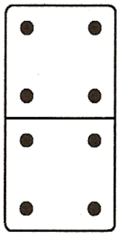
13



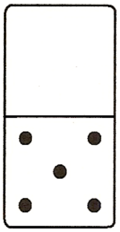
14



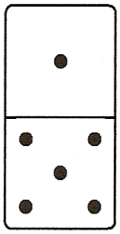
15



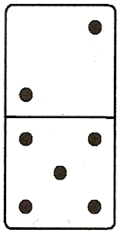
16



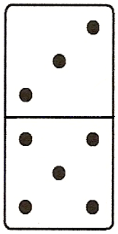
17



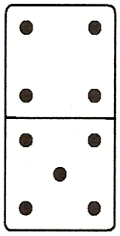
18



19

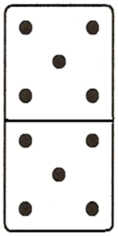


20

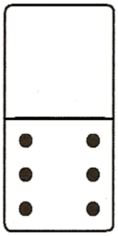


Appendix: Pictures of Dominos

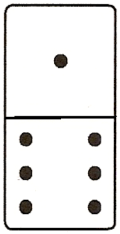
21



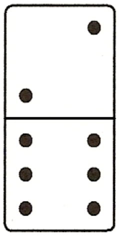
22



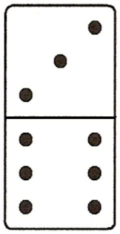
23



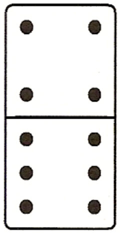
24



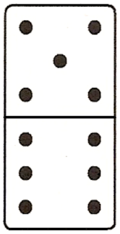
25



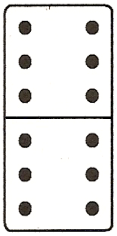
26



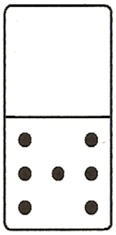
27



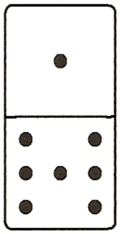
28



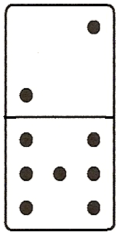
29



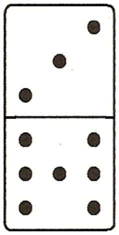
30



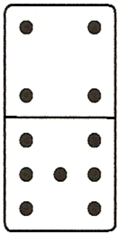
31



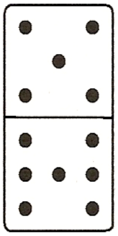
32



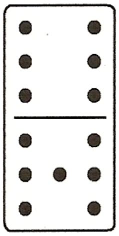
33



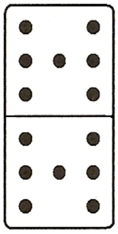
34



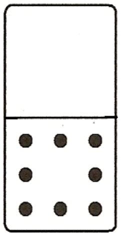
35



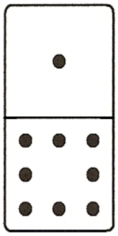
36



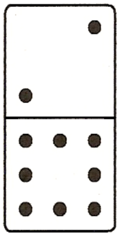
37



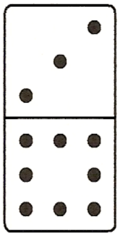
38



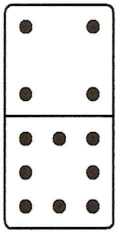
39



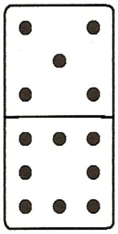
40



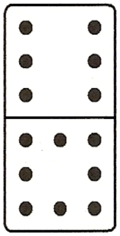
41



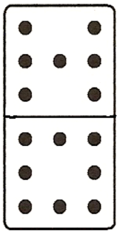
42



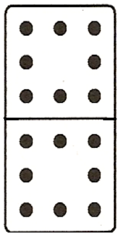
43



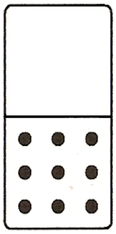
44



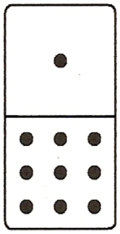
45



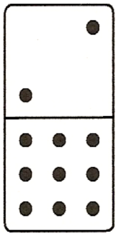
46



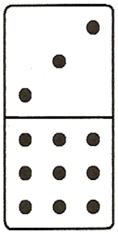
47



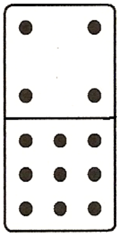
48



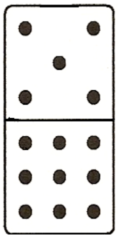
49



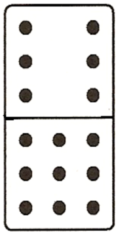
50



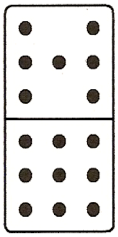
51



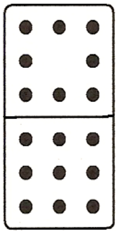
52



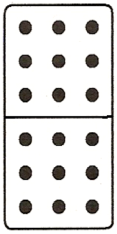
53



54



55



Index